Name ______________________

[redacted]

Exam 1 (5 problems, maximum 100 points)

[redacted]

You have 50 minutes to complete this exam. The exam is closed-book, closed-note.

Unless otherwise stated, show all of your work; heroic simplification is unnecessary. Full credit may not be given for an answer alone, and partial credit may be given for facts relevant to the solution. Multiple answers for any problem earn **zero** credit.

Do not unstable pages. **Loose pages will be ignored**.

Good luck!

1. (a) (10 pts) Complete the definition of independence: events E and F are **independent** if and only if ...

$P(E|F) = P(E).$

or $P(E \cap F) = P(E) \cap P(F)$

(b) (10 pts) Suppose that the events A, B, and C have the following properties:

$$\mathbb{P}(A) = 0.6 \quad \mathbb{P}(B) = 0.5 \quad \mathbb{P}(C) = 0.2 \quad \mathbb{P}(B|A) = 0.4$$

A and C are disjoint, B and C are independent. Then what is $\mathbb{P}(A \cup B \cup C)$?

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

$$= 0.6 + 0.5 + 0.2 - 0.24 - 0 - 0.1 + 0 = 0.96.$$

$P(A \cap B) = P(B|A)\,P(A) = 0.4 \cdot 0.6 = 0.24$

$P(A \cap C) = P(\phi) = 0$

$P(B \cap C) = P(B)\,P(C) = 0.5 \times 0.2 = 0.1$

$P(A \cap B \cap C) = P(\phi) = 0$

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2. Suppose that a blood test is 99% effective in detecting a disease given that the disease is present, i.e., $\mathbb{P}(T|D) = .99$, where T is the event that the test is positive and D is the event of the disease being present. Suppose also that the test also has a 2% chance of being positive when the disease is not present, and that 5% of the population actually has the disease.

(a) (10 pts) What is one of the relevant Bayes formulas for $\mathbb{P}(D|T)$?

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{P(T|D)\,P(D)}{P(T|D)\,P(D) + P(T|D^c)\,P(D^c)}$$

$P(T|D^c) = 2\%$,

$P(D) = 5\%$, $P(D^c) = 95\%$

(b)(10 pts) What is $\mathbb{P}(D|T)$, the probability of actually having the disease given that the test is positive? Leaving your answer unsimplified is fine, and approximating is a good idea.

$$P(D|T) = \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.95 \times 0.02}$$

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3. Suppose you draw 13 cards uniformly at random from a standard 52-card deck (4 suits: Hearts, Diamonds, Spades, and Clubs; and 13 denominations: Ace, 2, 3, ..., 10, J, Q, King). What is the probability that you at least have both the Ace and King from one suit? That is, if we define A_1 as the event of having Ace and King of Hearts, and A_2 ..., and A_4 having Ace and King of Clubs, then what is $\mathbb{P}(A_1 \cup A_2 \cup A_3 \cup A_4)$?

$P(A_1) = P(A\heartsuit, K\heartsuit \text{ in hand})$

$P(A_1 \cup A_2 \cup A_3 \cup A_3)$

$= P(A_1) + \cdots P(A_4).$

$- P(A_1 \cap A_2) - \cdots - P(A_3 \cap A_4)$

$+ P(A_1 \cap A_2 \cap A_3) + \cdots + P(A_2 \cap A_3 \cap A_4)$

$- P(A_1 \cap A_2 \cap A_3 \cap A_4)$

$\binom{4}{1} \; P(A_1) = \binom{4}{1} \dfrac{\binom{50}{11}}{\binom{52}{13}}$

$$= \binom{4}{1} \frac{\binom{50}{11}}{\binom{52}{13}} - \binom{4}{2} \frac{\binom{48}{9}}{\binom{52}{13}} + \binom{4}{3} \frac{\binom{46}{7}}{\binom{52}{13}} - \binom{4}{4} \frac{\binom{44}{5}}{\binom{52}{13}}$$

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4. (20 pts) Let X be a random variable with cumulative distribution function F given by

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{4}, & 0 \le x < 1, \\ \frac{1}{2}, & 1 \le x < 2, \\ \frac{x+6}{12}, & 2 \le x < 3, \\ 1, & 3 \le x. \end{cases}$$

Compute

$\mathbb{P}(X \le a) = F(a)$.

(a) (5 pts) $\mathbb{P}(X = 3)$,

$= \mathbb{P}(X \le 3) - \mathbb{P}(X < 3)$

$= F(3) - F(3^-) = 1 - \frac{9}{12} = \frac{1}{4}$

$F(3^-) = \left(\frac{x+6}{12}\right)\Big|_{x=3^-}$

(b) (5 pts) $\mathbb{P}(1 \le X < 3)$,

$F(1^-) = \frac{1}{4}$.

$= \mathbb{P}(X < 3) - \mathbb{P}(X < 1)$

$= F(3^-) - F(1^-) = \frac{9}{12} - \frac{1}{4} = \frac{1}{2}$

(c) (5 pts) $\mathbb{P}(X \ge 1/2)$,

$= 1 - \mathbb{P}(X < \frac{1}{2}) = 1 - F(\frac{1}{2}^-) = 1 - \frac{1}{4} = \frac{3}{4}$.

(d) (5 pts) $\mathbb{P}(X \le 7)$.

$= F(7) = 1$

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5. Team I and II play a series of 8 games. Team I wins each game with probability 0.6, independent of the outcomes of the other games. Let X be the total number of games played that team I won

(a) (5 pts) Find possible values of X. (5 pts) Find the PMF of X.

$\{0, 1, 2, \cdots, 8\}$.

$p(i) = \mathbb{P}(X=i) = \binom{8}{i}(0.6)^i(0.4)^{8-i}$

$i = 0, 1, 2, \cdots, 8$.

(b) (5 pts) Find the expectation of X.

$\mathbb{E}[X] = np = 8 \cdot 0.6 = 4.8$

(c) (5 pts) Find the probability that team II wins at least 6 games.

team II wins ≥ 6 games

= team I loses ≥ 6 games

= team I wins ≤ 2 games. $= \mathbb{P}(X \leq 2)$.

$\mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2)$

$= (0.4)^8 + 8(0.6)(0.4)^7 + \binom{8}{2}(0.6)^2(0.4)^6$

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Total		________
Problem	1	________
Problem	2	________
Problem	3	________
Problem	4	________
Problem	5	________