MATH 441 Section B/F13, Version A EXAM 1

Date: 09/23/2015 NAME: Solution

Problem 1 25 pt

Problem 2 25 pt

Problem 3 25 pt

Problem 4 25 pt

Problem 1. Fill out the boxes. No justification is necessary.

(1) True or false: There exists an initial value problem with infinitely many solutions.

True	5 (2pt)

(2) Write down two distinct solutions of

$$\begin{cases} \dfrac{dy}{dt} = y^{1/12} \\ y(0) = 0 \end{cases}$$

$y_1(t) = 0$ for all t	$y_2(t) = \left(\dfrac{11t}{12}\right)^{\frac{12}{11}}$	5 (2pts)

(3) Determine linearity/non-linearity of the following ODEs.

$(A)\ 5y'(t) + 2y(t) + 7\sqrt{y(t)} = 0 \qquad (B)\ e^t y'''(t) + t^{40} y''(t) + \sin(t) y(t) = 0$

(A): Non-linear	(B): Linear	5 (3pts)

(4) Let $p(t)$ and $g(t)$ be continuous functions on $\mathbb{R}$. Write down the unique solution of

$$\begin{cases} y'(t) + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

$y(t) =$

$$e^{-\int_{t_0}^{t} p(s)ds}\left(\int_{t_0}^{t} e^{\int_{t_0}^{s} p(u)du} g(s)\,ds + y_0\right)$$

or $\dfrac{1}{\mu(t)}\left(\int_{t_0}^{t} \mu(s) g(s)\,ds + y_0\right)$, $\mu(t) = e^{\int_{t_0}^{t} p(s)ds}$

5 (3pts)

(5) Write the equivalent integral equation for the following IVP:

$$\begin{cases} y'(t) = f(t, y), \\ y(t_0) = y_0 \end{cases}$$

$$\phi(t) = \int_{t_0}^{t} f(s, \phi(s))\,ds + y_0$$

5 (2pts)

Problem 2. Consider the IVP

$$\begin{cases} (5+t^2)y'(t) + 2ty = 15t \\ y(0) = 2 \end{cases}$$

Find the unique solution $y(t)$ and compute $\lim_{t\to\infty} y(t)$. (25pts)

Method 1: Solve linear ODE: $y'(t) + \frac{2t}{5+t^2} y = \frac{15t}{5+t^2}$

(5pt) Integrating factor $\mu(t) = \exp\left(\int_0^t \frac{2s}{5+s^2}\,ds\right) = e^{(\ln(5+t^2) - \ln 5)} = \frac{5+t^2}{5}$

(10pt) Solution is $y(t) = \frac{5}{5+t^2}\left(\int_0^t \frac{1}{5}(5+s^2)\cdot\frac{15s}{(5+s^2)}\,ds + c\right)$

plug in $y(0)=2$,

$2 = c$

$\Rightarrow y(t) = \frac{5}{5+t^2}\left(\int_0^t 3s\,ds + 2\right) = \frac{15t^2}{2(5+t^2)} + \frac{10}{5+t^2}$

(5pt) $= \frac{15t^2+20}{2(5+t^2)}$

(5pt) when $t\to\infty$, $y(t) \to \frac{15}{2}$. So $\lim_{t\to\infty} y(t) = \frac{15}{2}$

Method 2: Solve exact ODE: $\underbrace{(5+t^2)}_{M}dy + \underbrace{(2ty - 15t)}_{N}dt = 0$

$M_t = 2t$, $N_y = 2t$ $\Rightarrow$ Exact. (5pt)

There exists a Ψ s.t. $\begin{cases} \Psi_y = 5+t^2 & ① \\ \Psi_t = 2ty - 15t & ② \end{cases}$

Integrate ①, $\Psi = (5+t^2)y + h(t)$. plug into ②

$\Psi_t = 2ty + h'(t) = 2ty - 15t$

$\Rightarrow h(t) = -\frac{15}{2}t^2$, $\Psi = (5+t^2)y - \frac{15}{2}t^2$ (5pt)

$\Rightarrow (5+t^2)y - \frac{15}{2}t^2 = c$ is the general solution. (5pt) Plug in $y(0)=2$

$10 = c$ $\Rightarrow (5+t^2)y - \frac{15}{2}t^2 = 10 \Rightarrow y = \frac{15t^2+20}{2(5+t^2)}$ $\lim y = \frac{15}{2}$

Problem 3. Consider the differential equation

$$y'(t) + 1 = \frac{y}{t + 2y}$$

Find the general solution. (25pts)

Method 1: Exact ODE $\underbrace{(t+2y)}_{M}dy + \underbrace{(t+y)}_{N}dt = 0$

$M_t = 1,\ N_y = 1 \Rightarrow$ Exact. (5pt)

There exists a ψ s.t $\begin{cases} \psi_y = t+2y & ① \\ \psi_t = t+y & ② \end{cases}$

Integrate ①. $\psi = ty + y^2 + h(t)$. plug into ② (5pt)

$\psi_t = y + h'(t) = t + y \Rightarrow h(t) = \frac{t^2}{2} + c$

$\Rightarrow \psi = ty + y^2 + \frac{t^2}{2} + c$ (5pt)

Therefore $ty + y^2 + \frac{t^2}{2} = c$, Complete the square (5pt)

$(y + \frac{t}{2})^2 = c - \frac{t^2}{4}$, $y = -\frac{t}{2} \pm \sqrt{c - \frac{t^2}{4}}$ (5pt)

is the general solution.

Method 2: Solve homogeneous eqn: $y'(t) = -\frac{t+y}{t+2y} = -\frac{1+\frac{y}{t}}{1+\frac{2y}{t}}$

let $v = \frac{y}{t}$, then $y'(t) = v'(t)\cdot t + v(t)$ (5pt)

$v'\cdot t + v = -\frac{1+v}{1+2v} \Rightarrow v'\cdot t = -\frac{1+2v+2v^2}{1+2v}$ (5pt)

$\frac{(1+2v)dv}{1+2v+2v^2} = -\frac{dt}{t}$ $\quad \frac{1}{2}\ln(2v^2+2v+1) = \ln\frac{1}{t} + c$ (5pt)

$\Rightarrow 2v^2+2v+1 = t^{-2}C$, plug in $v = \frac{y}{t}$ $\quad y^2 + yt + \frac{t^2}{2} = c$ (5pt)

$\Rightarrow (y + \frac{t}{2})^2 = c - \frac{t^2}{4} \Rightarrow y = -\frac{t}{2} \pm \sqrt{c - \frac{t^2}{4}}$ (5pt)

or $y = \frac{-t \pm \sqrt{c - t^2}}{2}$

Problem 4. Let $f(t,y) = t^2 + \sqrt{16-y^2}$ for $R: -3 \le t \le 3, -4 \le y \le 4$, and undefined elsewhere. Consider

$$\begin{cases} y'(t) = f(t,y), \\ y(0) = 0 \end{cases}$$

(1) What is the maximum time interval $(-h, h)$ for which Picard iteration is valid? (12pts)
(2) On this time interval, set $\phi_0(t) = 0$, compute the first Picard iterate $\phi_1(t)$. (13pts)

(1). Compute the maximum value of $|f|$ inside R

$$M = \max_{\substack{-3 \le t \le 3 \\ -4 \le y \le 4}} \left| t^2 + \sqrt{16-y^2} \right| = 9 + 4 = 13. \quad (6pt)$$

$$h := \min\left\{\frac{b}{M}, a\right\} \qquad b = 4,\ a = 3.$$

$$= \min\left\{\frac{4}{13}, 3\right\} = \frac{4}{13}.$$

the interval is $\left(-\frac{4}{13}, \frac{4}{13}\right)$. (6pt)

(2).

$$\phi_1(t) = \int_0^t \left(s^2 + \sqrt{16-\phi_0(s)}\right) ds \quad (6pt)$$

$$= \int_0^t (s^2 + 4)\, ds = \frac{t^3}{3} + 4t \quad (7pt)$$