MATH 441 **VERSION A** **EXAM 2**

Date: 10/21/2015 NAME: Solution

Problem 1 __________ (25pts)

Problem 2 __________ (25pts)

Problem 3 __________ (25pts)

Problem 4 __________ (25pts)

Problem 1. Fill out the boxes. No justification is necessary.

(1) Write the ODE that describes the logistic growth with threshold model, clarify the range of your constants.

ODE: $\frac{dy}{dt} = -r(1-\frac{y}{T})(1-\frac{y}{k})y$, $r>0$, $0<T<k$ (5pts)

(2) Given $y_1(t)$, $y_2(t)$ are solutions of ODE: $y''(t)+p(t)y'(t)+q(t)y(t)=0$ for $\alpha<t<\beta$. Answer the True/False questions:

(a) $y_1(t)$, $y_2(t)$ form a fundamental set of solutions if and only if $W[y_1,y_2](t)\neq 0$ for all $\alpha<t<\beta$.
(b) $y_1(t)$, $y_2(t)$ form a fundamental set of solutions if and only if there exists $\alpha<t_0<\beta$ such that $W[y_1,y_2](t)\neq 0$.

(a) True (b) True (5pts)

(3) Let c_1, c_2 be arbitrary constants. Answer the True/False questions:

(a) Let $y_1(t)$, $y_2(t)$ be solutions of $y''(t)+p(t)y'(t)+q(t)y(t)=0$ for $t\in I$, then $c_1y_1(t)+c_2y_2(t)$ is the general solution of $y''(t)+p(t)y'(t)+q(t)y(t)=0$ in I.
(b) Let $y_1(t)$, $y_2(t)$ be solutions of $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$ for $t\in I$, then $c_1y_1(t)+c_2y_2(t)$ is also a solution of $y''(t)+p(t)y'(t)+q(t)y(t)=g(t)$ in I.

(a) False (b) False (5pts)

(4) Let $\{\Phi_n(t)\}$ be a sequence of continuous functions. At each point $t\in\mathbb{R}$, we know $\Phi_n(t)$ converges to a number and we denote by $\Phi(t)$. Then $\Phi(t)=\lim_{n\to+\infty}\Phi_n(t)$ is a function but not necessarily continuous. State the sufficient condition to guarantee the continuity of $\Phi(t)$.

$\Phi_n(t)$ converges to $\Phi(t)$ uniformly (5pts)

(5) Consider the ODE

$$y''(t)+2ty'(t)+t^2y(t)=0.$$

Let $y_1(t)$ and $y_2(t)$ be solutions such that $W(y_1,y_2)(0)=10$. Then

$W(y_1,y_2)(t)=$ $10e^{-t^2}$ (5pts)

Problem 2. Solve the IVP problem

$$\begin{cases} y''(t) + 2y'(t) + 2y(t) = 0 \\ y(0) = 2, y'(0) = -2 \end{cases}$$

(25pt)

Solution: Char. eqn. $r^2 + 2r + 2 = 0$

$$r_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$y_1 = e^{-t+it} = e^{-t}(\cos t + i\sin t)$$

$$y_2 = e^{-t-it} = e^{-t}(\cos t - i\sin t) \qquad (10pt)$$

$$u(t) = e^{-t}\cos t, \quad v(t) = e^{-t}\sin t \qquad (5pt)$$

$$W[u,v] = \begin{vmatrix} e^{-t}\cos t & e^{-t}\sin t \\ -e^{-t}\cos t - e^{-t}\sin t & -e^{-t}\sin t + e^{-t}\cos t \end{vmatrix} \qquad (5pt)$$

$$= e^{-2t} \neq 0$$

$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$ is the general soln

$$y(0) = c_1 = 2, \qquad y'(0) = -c_1 + c_2 = -2$$

$$\Rightarrow \quad c_2 = 0 \qquad (5pt)$$

$$\Rightarrow \quad y = 2e^{-t}\cos t$$

Problem 3.
Let a, b, c be real constants such that $u(t) = e^{-2t}\cos t$, $v(t) = e^{-2t}\sin t$ solve the ODE

$$ay''(t) + by'(t) + cy(t) = 0.$$

Given $a = 2$, determine b and c. (25pts)

Solution 1: $u(t) = e^{-2t}\cos t$, $v(t) = e^{-2t}\sin t$

$\Rightarrow$ Corresponding complex solns are

$y_1(t) = e^{-2t}(\cos t + i\sin t) = e^{-2t+it}$

$y_2(t) = e^{-2t}(\cos t - i\sin t) = e^{-2t-it}$ (10pt)

corresponding roots are (5pt)

$r_1 = -2+i$, $r_2 = -2-i$

$\Rightarrow$ char. eqn $a(r-r_1)(r-r_2) = 0$ (5pt)

$a(r^2 - (r_1+r_2)r + r_1r_2) = 0$

$r_1 + r_2 = -4$, $r_1 r_2 = 4+1 = 5$

$a(r^2 + 4r + 5) = 0$. $a = 2 \Rightarrow b = 4a = 8$, $c = 5a = 10$.

$b = 8$. $c = 10$ (5pt)

Solution 2: Plug $u(t) = e^{-2t}\cos t$ in $L[u] = 0$ (5pt)

$u' = -2e^{-2t}\cos t - e^{-2t}\sin t$ (5pt)

$u'' = 3e^{-2t}\cos t + 4e^{-2t}\sin t$

So $au'' + bu' + cu = e^{-2t}(3a\cos t + 4a\sin t - 2b\cos t - b\sin t + c\cos t)$

$= 0$ (5pt)

$\Rightarrow (3a - 2b + c)\cos t = 0$ $(4a - b)\sin t = 0$ (5pt)

$\Rightarrow b = 4a$, $c = 2b - 3a = 5a$

$\Rightarrow b = 8$, $c = 10$ (5pt)

Problem 4. (1) **Use the method of reduction of order** to find two solutions $y_1(t)$, $y_2(t)$ of the following ODE to form a fundamental set of solutions. (15pt)

$$y''(t) + y'(t) + \frac{1}{4}y = 0.$$

(2) Verify $y_1(t)$ and $y_2(t)$ is indeed a fundamental set. (5pt)
(3) Write the general solution of the ODE. (5pt)

Solution (1) char. eqn: $r^2 + r + \frac{1}{4} = 0$

$$\left(r+\frac{1}{2}\right)^2 = 0, \quad r = -\frac{1}{2}$$

$y_1 = e^{-\frac{1}{2}t}$, Need to find another soln (5pt)

let $y = v(t)\, y_1(t) = v(t)\, e^{-\frac{1}{2}t}$ s.t $L[vy_1] = 0$

$$(vy_1)' = v' e^{-\frac{1}{2}t} - \frac{1}{2}v(t)\, e^{-\frac{1}{2}t}$$

$$(vy_1)'' = v'' e^{-\frac{1}{2}t} - \frac{1}{2}v' e^{-\frac{1}{2}t} - \frac{1}{2}v' e^{-\frac{1}{2}t} + \frac{1}{4}v e^{-\frac{1}{2}t}$$

$$= v'' e^{-\frac{1}{2}t} - v' e^{-\frac{1}{2}t} + \frac{1}{4}v e^{-\frac{1}{2}t}$$

$$L[ve^{-\frac{1}{2}t}] = v'' e^{-\frac{1}{2}t} - v' e^{-\frac{1}{2}t} + \frac{1}{4}v e^{-\frac{1}{2}t} + v' e^{-\frac{1}{2}t} - \frac{1}{2}v(t)\, e^{-\frac{1}{2}t} + \frac{1}{4}v e^{-\frac{1}{2}t}$$

$$= v'' e^{-\frac{1}{2}t} \qquad (5pt)$$

$L[ve^{-\frac{1}{2}t}] = 0 \Rightarrow v'' = 0, \Rightarrow v' = c_1, \; v = c_1 t + c_2$

$y = (c_1 t + c_2)\, e^{-\frac{1}{2}t}$, let $c_1 = 1, c_2 = 0$ $\; y_2 = t e^{-\frac{1}{2}t}$ (5pt)

(2) $$W[y_1, y_2] = \begin{vmatrix} e^{-\frac{1}{2}t} & t e^{-\frac{1}{2}t} \\ -\frac{1}{2}e^{-\frac{1}{2}t} & e^{-\frac{1}{2}t} - \frac{1}{2}t e^{-\frac{1}{2}t} \end{vmatrix} = e^{-t} \neq 0 \text{ for all } t.$$

So $y_1(t) = e^{-\frac{t}{2}}$, $y_2(t) = t e^{-\frac{1}{2}t}$ is a fund. set.

(3) $y = c_1 e^{-\frac{t}{2}} + c_2 t e^{-\frac{1}{2}t}$