MATH 476/567 ACTUARIAL RISK THEORY FALL 2016
PROFESSOR WANG

Homework 4 (max. points = 100)
Due at the beginning of class on Tuesday, October 25, 2016

You are encouraged to work on these problems in groups of no more than 4. However, each student must hand in her/his own answer sheet. Please show your work enough to show that you understand how to do the problem -and circle your final answer. Full credit can only be given if the answer and approach are appropriate. Please give answers to two decimal places -e.g., xx.xx% and $xx,xxx.xx.

1. $X(t)$ follows an arithmetic Brownian motion with a drift factor of 0.35 and a volatility of 0.43. Given that $X(4) = 2$.

(a) Find the mean of the normal distribution $X(13)$.
(b) Find the standard deviation of the normal distribution $X(13)$.
(c) What is the probability that $X(13) > 9$?

Solution: We know that

$$X(13) - X(4) \sim N(0.35 * 9, 0.43^2 * 9) = N(3.15, 1.6641)$$

Since $X(4) = 2$, then we obtain

$$X(13) \sim N(5.15, 1.6641)$$

. Hence the mean of the normal distribution $X(13)$ is 5.15. The standard deviation of the normal distribution $X(13)$ is $\sqrt{1.6641} = 1.29$. The probability of $X(13) > 9$ is

$$P(X(13) > 9) = P(Z > \frac{9 - 5.15}{1.29}) = P(Z > 2.9845) = 0.0014$$

2. Suppose that two stocks, A and B, have prices that each follow an arithmetic Brownian motion process, specifically:

$$dA = 3dt + 8dZ_A \quad dB = 4dt + 6dZ_B$$

The current price of stock A is 40, and the current price of stock B is 50. The two stocks are positively correlated; their correlation coefficient is $\rho = 0.50$. You currently have a portfolio consisting of two shares of stock A, and one share of stock B. Find the 85th percentile of your portfolio value one year from now, i.e., the value for which your portfolio value, one year from now, has a 15% probability of being greater than.

Solution: From the question, we know that

$$A(1) \sim N(40 + 3 * 1, 8^2 * 1) = N(43, 64)$$

$$B(1) \sim N(50 + 4 * 1, 6^2 * 1) = N(54, 36)$$

Therefore

$$2A(1) \sim N(86, 256)$$

Hence

$$2A(1) + B(1) \sim N(86 + 54, 256 + 36 + 2 * 0.5 * \sqrt{36 * 256}) = N(140, 388)$$

Then the 85th percentile of your portfolio value one year from now is $140 + \sqrt{388} * \Phi^{-1}(0.85) = 160.415$.

3. Suppose you are considering the purchase of a one-year option that will pay you either \$0 or \$1,250 one year from now, depending upon the price of a share of stock S (i.e., a ?binary? option). You select the following model of the stock price over the next year:

$$dS = 3dt + 8dZ$$

The current price of the stock is 150. The binary option will pay \$1,250 only if $S > 155$ one year from now. The effective annual interest rate is 8%. Calculate the present value of the expected option payoff.

Solution: Since $S(0) = 150$, then

$$S(1) \sim N(150 + 3, 8^2) = N(153, 64)$$

. Therefore

$$P(S(1) > 155) = P(Z > \frac{155 - 153}{8}) = P(Z > 0.25) = 0.4013$$

Hence the present value of the expected option payoff is

$$1250 * P(S(1) > 155) * \frac{1}{1.08} = 464.47$$

4. The price of a stock follows an Ornstein-Uhlenbeck process. The reversion factor for this process is 0.0906, and the volatility factor is 0.63. At some time t, $dt = 1$, $dZ(t) = 0.9356$, and $dX(t)$, the instantaneous change in the stock price, is 0.0215. The long-run equilibrium level is zero. What is the price of the stock at time t?

Solution: We know that

$$dX(t) = \lambda(\alpha - X(t))dt + \sigma dZ(t)$$

Since we know $\lambda = 0.0906$, $\sigma = 0.63$, $\alpha = 0$, $dt = 1$, $dZ(t) = 0.9356$, and $dX(t) = 0.0215$, we can obtain

$$X(t) = 6.27$$

5. The current price of a stock is 150. The stock price follows a geometric Brownian motion with drift rate of 10% per year and variance rate of 9% per year. Calculate the probability that two years from now the price of the stock will exceed 235.

Solution: We want

$$P(S(2) > 235) = P(ln(\frac{S(2)}{S(0)}) > ln(\frac{235}{150}))$$

Since we know

$$ln(\frac{S(2)}{S(0)}) \sim N((\alpha - 0.5\sigma^2)t, \sigma^2 t) = N(0.11, 0.18)$$

hence

$$P(S(2) > 235) = P(Z > \frac{ln(\frac{235}{150}) - 0.11}{\sqrt{0.18}}) = P(Z > 0.7989) = 0.2122$$

6. Use Itô's Lemma to find $d(2S^2(t))$ given

(1) $S(t)$ follows an arithmetic Brownian motion: $dS(t) = \alpha dt + \sigma dZ(t)$;

Solution:

$$\begin{aligned} d(2S^2(t)) &= 4SdS + 2(dS)^2 \\ &= 4S(\alpha dt + \sigma dZ(t)) + 2\sigma^2 dt \\ &= (4S\alpha + 2\sigma^2)dt + 4S\sigma dZ(t) \end{aligned}$$

(2) $S(t)$ follows an geometric Brownian motion: $\frac{dS(t)}{S(t)} = \alpha dt + \sigma dZ(t)$

Solution:

$$\begin{aligned} d(2S^2(t)) &= 4SdS + 2(dS)^2 \\ &= 4S^2(\alpha dt + \sigma dZ(t)) + 2\sigma^2 S^2 dt \\ &= (4S^2\alpha + 2\sigma^2 S^2)dt + 4S^2\sigma dZ(t) \end{aligned}$$

(3) $S(t)$ follows an a mean-reverting process (Ornstein-Uhlenbeck): $dS(t) = \lambda(\alpha - S(t))dt + \sigma dZ(t)$

Solution:

$$\begin{aligned} d(2S^2(t)) &= 4SdS + 2(dS)^2 \\ &= 4S(\lambda(\alpha - S)dt + \sigma dZ(t)) + 2\sigma^2 dt \\ &= (4S\lambda(\alpha - S) + 2\sigma^2)dt + 4S\sigma dZ(t) \end{aligned}$$

7. Consider two non-dividend-paying assets X and Y. There is a single source of uncertainty which is captured by a standard Brownian motion $Z(t)$. The prices of the assets satisfy the stochastic differential equations

$$\frac{dX(t)}{X(t)} = 0.06dt + 0.11dZ(t), \quad \frac{dY(t)}{Y(t)} = Adt + BdZ(t),$$

where A and B are constants. You are also given:

(i) $d[\ln Y(t)] = \mu dt + 0.085dZ(t)$;

(ii) The continuously compounded risk-free interest rate is 3%.

Determine A.

Solution: As shown in the notes, we can obtain

$$d[\ln Y(t)] = (A - 0.5B^2)dt + BdZ(t)$$

Therefore $B = 0.085$ and $A - 0.5B^2 = \mu$, which is useless. Since they have the same source of uncertainty, they have the same Sharpe ratios, that is

$$\frac{0.06 - 0.03}{0.11} = \frac{A - 0.03}{0.085}$$

Therefore $A = 0.0532$.

8. Consider an arbitrage-free securities market model, in which the risk-free interest rate is constant. There are two non-dividend-paying stocks whose price processes are

$$S_1(t) = S_1(0)e^{0.1t+0.2Z(t)}, \quad S_2(t) = S_2(0)e^{0.125t+0.3Z(t)}$$

where $Z(t)$ is a standard Brownian motion and $t \geq 0$. Determine the continuously compounded risk-free interest rate.

Solution: We assume

$$\frac{dS_1(t)}{S_1(t)} = \alpha_1 dt + \sigma_1 dZ(t)$$

and

$$\frac{dS_2(t)}{S_2(t)} = \alpha_2 dt + \sigma_2 dZ(t)$$

Then

$$S_1(t) = S_1(0)e^{(\alpha_1 - 0.5\sigma_1^2)t + \sigma_1 Z(t)}$$

and

$$S_2(t) = S_2(0)e^{(\alpha_2 - 0.5\sigma_2^2)t + \sigma_2 Z(t)}$$

According to the question, we can obtain $\sigma_1 = 0.2$, $\alpha_1 = 0.12$, $\sigma_2 = 0.3$ and $\alpha_2 = 0.17$. Since they have the same source of uncertainty, they have the same Sharpe ratios, that is

$$\frac{\alpha_1 - r}{\sigma_1} = \frac{\alpha_2 - r}{\sigma_2}$$

Therefore $r = 0.02$.

9. A call option on a stock has elasticity of 3.1. The continuously compounded risk-free rate is 2.3% and the Black-Scholes price volatility of the call is 0.12. Suppose that the risk premium on the stock is 77% of the stock volatility. Find the expected annual continuously compounded return on the option.
Solution: Since the Sharpe ratio of a call on the stock is equal to Sharpe ratio of the underlying stock, then the Sharpe ratio of the call is $\frac{\alpha - r}{\sigma_{stock}} = 0.77$. Since the Sharpe ratio of the call is also equal to $\frac{\gamma - r}{\sigma_{call}} = 0.77$, therefore

$$\gamma = r + 0.77 * \sigma_{call} = 0.1154$$

10. A stock has volatility of 23%. The annual continuously compounded risk-free is 4% and the annual continuously compounded return on the stock is 9%. What is the Sharpe ratio of a put on the stock?
Solution: Since the Sharpe ratio of a put on the stock is equal to negative Sharpe ratio of the underlying stock, then

$$Sharpe\ ratio = -\frac{\alpha - r}{\sigma_{stock}} = -\frac{0.09 - 0.04}{0.23} = -0.2174$$

Additional Problems for Math 567 Students (max. points = 20)

11. Solve the Ornstein-Uhlenbeck process from the stochastic differential equation

$$dX(t) = \lambda(\alpha - X(t))dt + \sigma dZ(t).$$

Hint: change variable $Y(t) = X(t) - \alpha$.
Solution: We assume $Y(t) = X(t) - \alpha$. In this case, we obtain that

$$dY(t) = 1dX(t) = -\lambda Y(t)dt + \sigma dZ(t)$$

Or

$$dY(t) + \lambda Y(t)dt = \sigma dZ(t)$$

This can be written as

$$d[e^{\lambda t}Y(t)] = e^{\lambda t}\sigma dZ(t)$$

Hence integrating from 0 to t we obtain

$$e^{\lambda t}Y(t) - Y(0) = \sigma \int_0^t e^{\lambda s} dZ(s)$$

Writing the answer in terms of X we find

$$X(t) = X(0)e^{-\lambda t} + \alpha(1 - e^{-\lambda t}) + \sigma \int_0^t e^{-\lambda(t-s)} dZ(s)$$

12. $X(t)$ is an Ornstein-Uhlenbeck process defined by $dX(t) = 2[4-X(t)]dt+8dZ(t)$, where $Z(t)$ is a standard Brownian motion. Let $Y(t) = \frac{1}{X(t)}$. You are given that $dY(t) = \alpha(Y(t))dt+\beta(Y(t))dZ(t)$ for some functions $\alpha(y)$ and $\beta(y)$. Determine $\alpha(0.5)$.

Solution:

$$\begin{aligned} dY(t) &= -\frac{1}{X(t)^2}dX(t) + \frac{1}{X(t)^3}(dX(t))^2 \\ &= -Y(t)^2[2(4-\frac{1}{Y(t)})dt + 8dZ(t)] + 64Y(t)^3dt \\ &= (64Y(t)^3 - 8Y(t)^2 + 2Y(t))dt - 8Y(t)^2dZ(t) \end{aligned}$$

Therefore

$$\alpha(y) = 64y^3 - 8y^2 + 2y$$

and hence

$$\alpha(0.5) = 7$$