MATH 461 PROBABILITY THEORY SPRING 2017
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Solution 1

1. Let $X = \{1, 2, 3, 4, 5\}$. The sample space S consists of all finite sequences $(x_1, \ldots, x_n)$, where $x_1, \ldots, x_{n-1} \in X$ and $x_n = 6$, as well as infinite sequences with values in X.

E_n is the set of all sequences of length n in S, and $(\cup_{n=1}^{\infty} E_n)^c$ is the set of all infinite sequences in S, i.e., the set of infinite sequences with values in X.

2.

$$\begin{aligned} E =& \{(1,2),(1,4),(1,6),(2,1),(2,3),(2,5),\\ &(3,2),(3,4),(3,6),(4,1),(4,3),(4,5),\\ &(5,2),(5,4),(5,6),(6,1),(6,3),(6,5)\}\\ F =& \{(2,1),(2,2),(2,3),(2,4),(2,5),(2,6),\\ &(1,2),(3,2),(4,2),(5,2),(6,2)\}\\ G =& \{(1,4),(2,3),(3,2),(4,1)\} \end{aligned}$$

(a) Meaning of EF: The sum is odd *and* at least one of the dice lands on 2, that is

$$EF = \{(2,1),(2,3),(2,5),(1,2),(3,2),(5,2)\}.$$

(b) Meaning of $E \cup F$: The sum is odd *or* at least one the dice lands on 2.

(c) Meaning of FG: At least one of the dice lands on 2 and the sum equals 5, i.e., $FG = \{(2,3),(3,2)\}$.

(d) Meaning of EF^c: The sum is odd and none of the dice lands on 2.

(e) Meaning of EFG: The sum is odd and one of the dice lands on 2 and the sum is five, i.e., $EFG = FG$.

3. (a) There are 2^5 possible outcomes (two outcomes per component, use generalized counting principle).

(b) Represent an outcome as a binary number, e.g., 00101 means that components 3 and 5 work, and components 1, 2, and 4 do not work. Then

$$\begin{aligned} W =& \{11000, 11001, 11010, 11011, 11100, 11101, 11110, 11111,\\ &00110, 00111, 01110, 01111, 10110, 10111, 10101\}. \end{aligned}$$

(c) There are $2^3 = 8$ outcomes in A.

(d) $AW = \{11000, 11100\}$.

4. Number of possible area codes: $8 \cdot 2 \cdot 9 = 144$ (Generalized counting principle: There are eight choices for the first digit, two for the second digit, and nine for the third digit.)

Number of area codes starting with a 4: $1 \cdot 2 \cdot 9 = 18$

5. (a) $5! = 120$ (five distinct letters)

(b) $\frac{7!}{2! \cdot 2!} = 1260$ (seven letters, including two Ps and two Os)

(c) $\frac{11!}{4! \cdot 4! \cdot 2!} = 34,650$ (eleven letters, including four Is, four Ss, and two Ps)

(d) $\frac{7!}{2!\cdot 2!} = 1260$ (seven letters, two As, two Rs)

6. (a) 6! (b) 3!3!2! (c) 4!3!

7. (a) This is equivalent to choosing k positions out of n positions where the value will be 1 and 0 for the rest. Thus the answer is $\binom{n}{k}$.
(b) The answer is

$$\sum_{i=k}^{n} \binom{n}{i}$$

as the sum can be any number from k to n.

8. This is equivalent to in how many ways we can write $x_1 + x_2 + \cdots + x_r = n$ where $x_i \geq 2$ for all i.

Given a sequence $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_r = n$ and $x_i \geq 1$ for all i, write $y_i = x_i - 2$ for all i. Then we have $y_i \geq 0$ for all i and

$$y_1 + y_2 + \cdots + y_r = n - 2r.$$

Note that we can get back x_i's from y_i's by adding 2 to each of them. Thus the answer is same as in how many ways we can write

$$y_1 + y_2 + \cdots + y_r = n - 2r.$$

where $y_i \geq 0$ for all i. We did this in class and the answer is $\binom{n-2r+r-1}{r-1} = \binom{n-r-1}{r-1}$.

9. Following the hint we let E_n be the event that a 5 occurs on the n-th roll and no three or five occurs on the $(n-1)$ rolls up to that point. Then

$$\mathrm{P}(E_n) = \frac{4^{n-1} \cdot 1}{6^n},$$

as for n rolls total number of possible outcomes are 6^n and out of them E_n has 4^{n-1} many outcomes. Since we want the probability that a five comes first, this can happen at roll number one ($n = 1$), at roll number two ($n = 2$) or any subsequent roll. Moreover, the events E_n are mutually exclusive. Thus the probability that a five comes first is given by

$$\mathrm{P}(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mathrm{P}(E_n) = \sum_{n=1}^{\infty} \frac{4^{n-1}}{6^n} = \sum_{n=1}^{\infty} \frac{2^n}{4 \cdot 3^n} = \frac{1}{4} \cdot \frac{2/3}{1 - 2/3} = \frac{1}{2}.$$

10. We want the probability that a given hand of bridge is void in *at least* one suit which means the hand could be void in more than one suit out of four suits. Let E_i be the event that the hand is void in the suit i for $i = 1, 2, 3, 4$ (*clubs, hearts, diamonds* and *spades*). Then the probability we want is $\mathrm{P}(\cup_{i=1}^{4} E_i)$ which we can calculate by using the inclusion-exclusion identity and the symmetry of the suits, given in this case by

$$\begin{aligned}\mathrm{P}(E_1 \cup E_2 \cup E_3 \cup E_4) &= \sum_{i=1}^{4} \mathrm{P}(E_i) - \sum_{i=1}^{3} \sum_{j>i} \mathrm{P}(E_i E_j) + \sum_{i=1}^{3} \sum_{j>i} \sum_{k>j} \mathrm{P}(E_i E_j E_k) \\ &= 4\mathrm{P}(E_1) - 6\mathrm{P}(E_1 E_2) + 4\mathrm{P}(E_1 E_2 E_3).\end{aligned}$$

Note there is no terms $\mathrm{P}(E_i E_j E_k E_l)$ (which is zero) since we must be dealt some cards. We have $\mathrm{P}(E_1) = \frac{\binom{39}{13}}{\binom{52}{13}}$. Similarly

$$\mathrm{P}(E_1 E_2) = \frac{\binom{26}{13}}{\binom{52}{13}} \text{ and } \mathrm{P}(E_1 E_2 E_3) = \frac{\binom{13}{13}}{\binom{52}{13}} = \frac{1}{\binom{52}{13}}.$$

Thus we get

$$\mathrm{P}(E_1 \cup E_2 \cup E_3 \cup E_4) = \frac{1}{\binom{52}{13}} \left(4\binom{39}{13} - 6\binom{26}{13} + 4 \right).$$