Lecture 4 Exact eqn & Integrating factors

Thm: Let M, N, M_y, N_x be cont. func, in region $R: \alpha < x < \beta,\ \gamma < y < \delta$. Then

$$M(x,y) + N(x,y)\,y' = 0 \qquad ①$$

is an exact diff eqn. in R iff

$$M_y(x,y) = N_x(x,y)$$

at each pt in R. That is there exists a fun. ψ satisfies $\frac{\partial \psi}{\partial x} = M$, $\frac{\partial \psi}{\partial y} = N$. iff. M, N satisfies $M_y(x,y) = N_x(x,y)$

Proof: ① exact $\Rightarrow$ $M_y(x,y) = N_x(x,y)$

or $\exists\, \psi$. s.t.

$\frac{\partial \psi}{\partial x} = M$. $\psi_y = N$

Integrate $\frac{\partial \psi}{\partial x}$ w.r.t to x. holding y const.

$$\psi(x,y) = \int_{x_0}^{x} \frac{\partial \psi}{\partial x}\,dx + h(y) = \int_{x_0}^{x} M(s,y)\,ds + h(y)$$

$\alpha < x_0 < \beta$ is a constant. h is an arbitrary diff'ble funct of y. playing the role of arbitrary const.

Now since $\psi_y = N$. we have ② satisfies

$$N = \frac{\partial \psi}{\partial y} = \int_{x_0}^{x} \frac{\partial M(s,y)}{\partial y}\,ds + h'(y)$$

$\Rightarrow$ $h'(y) = N(x,y) - \int_{x_0}^{x} \frac{\partial M(s,y)}{\partial y}\,ds$ is only a function of y

~~This is if we differentiate it w.r.t. x, it's 0.~~

$\Rightarrow$ It's der. w.r.t x is 0.

$$\Rightarrow \quad 0 = \frac{\partial}{\partial x}\left(W(y)\right) = \frac{\partial}{\partial x} N(x,y) - \frac{\partial}{\partial x}\left(\int_{x_0}^{x} \frac{\partial M(s,y)}{\partial y}\, ds \right)$$

$$= \frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y)$$

$$\Rightarrow \quad N_x(x,y) = M_y(x,y)$$

"$\Leftarrow$". Know $M_y(x,y) = N_x(x,y)$

WTS $\exists\ \psi(x,y)$ s.t. $\psi_x(x,y) = M(x,y)$

$\psi_y(x,y) = N(x,y)$. ✓

Consider $\psi(x,y) = \int_{y_0}^{y} N(x,s)\, ds + \int_{x_0}^{x} M(s,y_0)\, ds$

$$\psi_y(x,y) = N(x,y)$$

$$\psi_x(x,y) = \int_{y_0}^{y} N_x(x,s)\, ds + M(x,y_0)$$

Use $N_x = M_y$

$$= \int_{y_0}^{y} M_y(x,s)\, ds + M(x,y_0)$$

$$= M(x,y) - M(x,y_0) + M(x,y_0)$$

$$= M(x,y)$$

$\blacksquare$

Example 1. Solve diff. eqn. ③

$$(\underbrace{y\cos x + 2xe^{y}}_{M}) + (\underbrace{\sin x + x^2 e^{y} - 1}_{N})y' = 0$$

Show it's an exact eqn.

Soln: step 1 First check exact eqn: $M_y = N_x$ ✓

$M_y(x,y) = \cancel{-y\sin x + 2e^{y}}\ \cos x + 2xe^{y}$

$N_x(x,y) = \cos x + 2xe^{y}$

Step 2 Solve the eqn. By thm 1, there exists a $\psi(x,y)$.

$$\psi_x(x,y) = y\cos x + 2xe^{y} \quad ③$$

$$\psi_y(x,y) = \sin x + x^2 e^{y} - 1 \quad ④$$

Integrat ③ w.r.t x $\quad \psi(x,y) = y\sin x + x^2 e^{y} + h(y)$, ⑤

find $h(y)$, diff. ⑤ w.r.t y $\quad \psi_y(x,y) = \sin x + x^2 e^{y} + h'(y)$ ⑥

compare ⑥ with ④ $\quad h'(y) = -1$, $\Rightarrow h(y) = -y + c'$

c' can be omitted,

$$\psi(x,y) = y\sin x + x^2 e^{y} - y \underline{+c'}$$

$\Rightarrow$ $\psi(x,y) = c$ is the soln.

$y\sin x + x^2 e^{y} - y = c$. is the soln

Example 2. Check exact and solve diff. eqn

$$(3xy + y^2) + (x^2 + xy)y' = 0$$

Soln: Step 1. check $M_y = 3x + 2y$, $N_x = 2x + y$

$M_y \neq N_x$

Not an exact. eqn. (4)

· Step2: Seek for a ψ s.t.

$$\psi_x(x,y) = 3xy + y^2, \quad \psi_y(x,y) = x^2 + xy$$

$$\Rightarrow \quad \psi(x,y) = \frac{3}{2}x^2 y + xy^2 + h(y)$$

$$\psi_y = \frac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

$$\Rightarrow \quad h'(y) = -\frac{1}{2}x^2 - xy \qquad \text{not possible.!}$$

<u>Integrating Factors</u> Non-exact eqn $\Rightarrow$ exact eqn.

* Same trick as solving linear diff. eqn.

Given $\quad M(x,y) + N(x,y)\, y' = 0 \qquad (A)$

Multiply both sides by $\mu(x,y)$, it becomes

$$\mu(x,y) M(x,y) + N(x,y)\, \mu(x,y)\, y' = 0$$

Want to find $\mu(x,y)$ s.t the above eqn is exact!

If Only if $\quad (\mu M)_y = (\mu N)_x$

This gives $\quad \mu_y \cdot M + \mu M_y = \mu_x N + \mu N_x$

$$\Rightarrow \quad M\mu_y - N\mu_x + (M_y - N_x)\mu = 0 \qquad (7)$$

* This is a partial. diff. eqn. may have more than 1 soln

* It is not easiler to be solved than the original ODE. (x)

* Integrating factor is powerful in principal, but in practice can only be found in <u>special cases</u>

Special case is when μ is a function of x only (or y only)

Assume μ is only a function of x. then $\mu_y = 0$. $\mu_x = \frac{d\mu(x)}{dx}$

(7) becomes $N \frac{d\mu}{dx} = (M_y - N_x)\mu$

$$\Rightarrow \quad \frac{d\mu}{dx} = \frac{M_y - N_x}{N}\mu \quad \text{is only a func. of } x$$

then $\mu(x)$ can be found by integrating

Example 3. Find integrating factor for

$$(\underbrace{3xy + y^2}_{M}) + (\underbrace{x^2 + xy}_{N})y' = 0$$

Soln: $M_y = 3x + 2y$ $\quad M_y \neq N_x$. non-exact.

$N_x = 2x + y$

Step 1: Find $\mu(x, y)$.

$$\mu(x,y)(3xy + y^2) + \mu(x^2 + xy)\, y' = 0$$

$$\left(\mu(3xy + y^2)\right)_y = \left(\mu(x^2 + xy)\right)_x$$

$$\mu_y(3xy + y^2) + \mu(3x + 2y) = \mu_x(x^2 + xy) + \mu(2x + y)$$

$$\mu(x + y) = \mu_x(x^2 + xy) - \mu_y(3xy + y^2).$$

if $\mu_x = 0$, then $\mu(x+y) = -\mu_y(3xy + y^2)$

$$\frac{d\mu}{dy} = -\frac{\mu(x,y)(x+y)}{3xy + y^2}$$

Not easy to solve

If $\mu_y = 0$ then $\frac{\mu(x,y)(x+y)}{x^2 + xy} = \mu_x$

$\Rightarrow \frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow d\ln|\mu| = d\ln|x| \Rightarrow \mu(x) = \pm|x|$ 6

$\Rightarrow \mu(x) = x.$

Step 2. solve eq^n. multiply μ to both sides of original eq^n.

$$(3x^2y + y^2x) + (x^3 + x^2y)y' = 0$$

it's exact.

$$\frac{\partial}{\partial y}(3x^2y + y^2x) = 3x^2 + 2yx$$

$$\frac{\partial}{\partial x}(x^3 + x^2y) = 3x^2 + 2xy$$

Then there exists a ψ s.t.

$$\psi_x(x,y) = 3x^2y + xy^2$$

$$\psi_y(x,y) = x^3 + x^2y$$

Integrate w.r.t. x $\psi(x,y) = x^3y + \frac{1}{2}x^2y^2 + h(y)$. plug into

$$\psi_y = x^3 + x^2y + h'(y) = x^3 + x^2y$$

$\Rightarrow h'(y) = 0. \quad h(y) = c.$

$\Rightarrow$ Solution is $\psi(x,y) = x^3y + \frac{1}{2}x^2y^2 + c = c'$

$$x^3y + \frac{1}{2}x^2y^2 = c.$$