MATH 476/567 ACTUARIAL RISK THEORY FALL 2016
PROFESSOR WANG

Homework 6 (max. points = 100)
Due at the beginning of class on Thursday, December 1, 2016

You are encouraged to work on these problems in groups of no more than 4. However, each student must hand in her/his own answer sheet. Please show your work enough to show that you understand how to do the problem -and circle your final answer. Full credit can only be given if the answer and approach are appropriate. Please give answers to two decimal places -e.g., xx.xx% and $xx,xxx.xx.

1. You are to use a Black-Derman-Toy model to determine $F_{0,2}[P(2,3)]$, the forward price for time-2 delivery of a zero-coupon bond that pays 1 at time 3. In the Black-Derman-Toy model, each period is one year. The following effective annual interest rates are given:

$$r_d = 3\% \ , r_u = 6\% \ , r_{dd} = 2\% \ , r_{uu} = 8\%$$

Determine $100 \times F_{0,2}[P(2,3)]$.

2. Using the tree of the previous problem, compute the volatility yield in year 1 of a 3-year zero-coupon bond.

3. You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0 = 0.04, r_d = 0.02, r_u = 0.12, r_{dd} = 0.047, r_{du} = 0.072$. A year-3 caplet has a strike rate of 0.07. The caplet has a notional amount of $100. What is the current value of this caplet?

4. You are given the following short-term interest rates in a Black-Derman-Toy binomial tree: $r_0 = 0.05, r_d = 0.04, r_u = 0.09, r_{dd} = 0.06, r_{uu} = 0.18, r_{ddd} = 0.04, r_{uud} = 0.16$. A year-4 floorlet has a strike rate of 0.07. The floorlet has a notional amount of $100. What is the current value of this floorlet?

5. A Vasicek model for a short-rate is given by

$$dr = 0.2(0.1 - r)dt + 0.02dZ(t).$$

The Sharpe ratio for interest rate risk is $\phi = 0$. Determine the price of a zero-coupon bond with par value of $100 and that matures in 10 years if the risk-free annual interest rate is 8%.

6. You are using the Vasicek one-factor interest-rate model with the short-rate process calibrated as

$$dr(t) = 0.5[b - r(t)]dt + dZ(t).$$

The price of each zero-coupon bond in the Vasicek model follows an Itô process,

$$\frac{dP(r,t,T)}{P(r,t,T)} = \alpha(r,t,T)dt - q(r,t,T)dZ(t), t \le T.$$

You are given that $\alpha(0.05, 1, 3) = 0.03692644$. Find $\alpha(0.04, 0, 4)$.

7. You are given the following information:

(i) A one-year \$1 zero-coupon bond has a price of 0.8941.

(ii) A two-year \$1 zero-coupon bond has a price of 0.8703.

(iii) A three-year \$1 zero-coupon bond has a price of 0.8216.

The bond forward price is log-normally distributed with volatility $\sigma = 0.11$. Using the Black model, calculate the price of a one-year European call option which gives you the right to purchase a one-year \$1 zero-coupon bond for 0.9040.

8. The time-0 price of a 2-year zero-coupon bond with par value \$1 is \$0.8853. The time-0 price of a 5-year zero-coupon bond with par value \$1 is \$0.6657. A European put option that expires in 2 years enables you to sell a 3-year bond at expiration for the price of \$0.7799. The forward price of the bond is log-normally distributed and has a volatility of 0.33. Find the price of this put option.

9. A stock is currently selling for \$100. The annual continuously compounded yield is 0.03. The annual continuously compounded risk-free interest rate is 0.11, and the stock price volatility is 0.30. Consider a \$102-strike put with one year to expiration. Using the three draws from the uniform distribution on (0,1): 0.12, 0.87, and 0.50, compute the price of the put using the Monte Carlo valuation.

10. The price of a stock is to be estimated using simulation. It is known that:

(i) The time-t stock price S_t follows the lognormal distribution: $\ln(S_t/S_{t-1})$ is normal with mean $(\alpha - 0.5\sigma^2)t$ and variance $\sigma^2 t$.

(ii) $S_0 = 50$, $\alpha = 0.15$, and $\sigma = 0.30$.

The following are three uniform $(0, 1)$ random numbers:

$$0.9370, 0.0384, 0.6594.$$

Use each of these three numbers to simulate stock prices at the end of 4 months, 8 months, and 12 months. (1) Calculate the arithmetic mean and the geometric mean of the three simulated prices. (2) Find the payoff at expiration of an Asian call option with strike price of \$50 based on the arithmetic average of the simulated prices of (1).

Additional Problems for Math 567 Students (max. points = 20)

11. You are given:

(i) The true stochastic process of the short-rate is given by

$$dr(t) = (0.09 - 0.5r(t))dt + 0.3dZ(t),$$

where $Z(t)$ is a standard Brownian motion under the true probability measure.

(ii) The risk-neutral process of the short-rate is given by

$$dr(t) = (0.15 - 0.5r(t))dt + \sigma(r(t))d\tilde{Z}(t),$$

where $\tilde{Z(t)}$ is a standard Brownian motion under the risk-neutral probability measure.

(iii) $g(r, t)$ denotes the price of an interest-rate derivative at time t, if the short-rate at that time is r. The interest-rate derivative does not pay any dividend or interest.

(iv) $g(r(t), t)$ satisfies

$$dg(r(t), t) = \mu(r(t), g(r(t), t))dt - 0.4g(r(t), t)dZ(t).$$

Determine $\mu(r, g)$ in terms of g and r.

12. The short-rate process $r(t)$ in a Cox-Ingersoll-Ross model follows

$$dr(t) = (0.015 - 0.12r(t))dt + 0.10\sqrt{r(t)}dZ(t),$$

where $Z(t)$ is a standard Brownian motion under the true probability mea- sure. For $t \leq T$, let $P(r, t, T)$ denote the price at time t of a zero-coupon bond that pays \$1 at time T, if the short-rate at time t is r. You are given:

(i) The Sharpe ratio takes the form $\phi(r, t) = c\sqrt{r}$.

(ii) $\lim_{T\to\infty} \frac{\ln(P(r,0,T))}{T} = -0.1$ for each $r > 0$.

Find the constant c.