MATH 476/567 ACTUARIAL RISK THEORY FALL 2016
PROFESSOR WANG

Homework 5 (max. points = 100)
Due at the beginning of class on Tuesday, November 8, 2016

You are encouraged to work on these problems in groups of no more than 4. However, each student must hand in her/his own answer sheet. Please show your work enough to show that you understand how to do the problem -and circle your final answer. Full credit can only be given if the answer and approach are appropriate. Please give answers to two decimal places -e.g., xx.xx% and $xx,xxx.xx.

1. Consider the following information about an Asian call on a stock: The strike price is $100. The current stock price is $100. The time to expiration is one year. The stock price volatility is 30%. The annual continuously-compounded risk-free rate is 8%. The stock pays no dividends. The price is calculated using two-step binomial model where each step is 6 months in length.
   (a) Construct the binomial stock price tree including all possible arithmetic and geometric averages after one year.
   (b) What is the price of an Asian arithmetic average price call?
   (c) What is the price of an Asian geometric average price call?

Solution:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = 1.286766$$
$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = 0.841868$$
$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = 0.447165$$

(a)

$$\begin{tikzcd}[column sep=small]
 & & 165.5766 \quad AA = 147.1266 \quad GA = 145.9652 \\
 & 128.6766 \arrow[ur] \arrow[dr] & \\
 & & 108.3287 \quad AA = 118.5026 \quad GA = 118.0651 \\
100 \arrow[uur] \arrow[ddr] & & \\
 & & 108.3287 \quad AA = 96.2578 \quad GA = 95.4979 \\
 & 84.1868 \arrow[ur] \arrow[dr] & \\
 & & 70.8742 \quad AA = 77.5305 \quad GA = 77.2442
\end{tikzcd}$$

(b)

$$C = e^{-rT}((p^*)^2 * 47.1266 + (p^*)(1-(p^*)) * 18.5026) = 12.92$$

(c)

$$C^{'} = e^{-rT}((p^*)^2 * 45.9652 + (p^*)(1-(p^*)) * 18.0651) = 12.61$$

2. Consider the following information about an Asian put on a stock: The strike price is \$150. The current stock price is \$145. The time to expiration is one year. The stock price volatility is 25%. The annual continuously-compounded risk-free rate is 7.5%. The stock pays no dividends. The price is calculated using two-step binomial model where each step is 1 year in length.

(a) Construct the binomial stock price tree including all possible arithmetic and geometric averages after two years.
(b) What is the price of an Asian arithmetic average strike put?
(c) What is the price of an Asian geometric average strike put?

Solution:

$$u = e^{(r-\delta)h+\sigma\sqrt{h}} = 1.384030$$

$$d = e^{(r-\delta)h-\sigma\sqrt{h}} = 0.839457$$

$$p^* = \frac{e^{(r-\delta)h} - d}{u-d} = 0.4378235$$

(a)

$$\begin{array}{lllll}
 & & 277.7534 & AA = 239.2189 & GA = 236.0949 \\
 & 200.6844 & & & \\
 & & 168.4660 & AA = 184.5752 & GA = 183.8709 \\
145 & & & & \\
 & & 168.4660 & AA = 145.0936 & GA = 143.1988 \\
 & 121.7213 & & & \\
 & & 102.1798 & AA = 111.9505 & GA = 111.5233
\end{array}$$

(b)

$$C = e^{-rT}((p^*)(1-(p^*)) * (184.5752 - 168.4660) + (1-(p^*))^2 * (111.9505 - 102.1798)) = 6.071$$

(c)

$$C' = e^{-rT}((p^*)(1-(p^*)) * (183.8709 - 168.4660) + (1-(p^*))^2 * (111.5233 - 102.1798)) = 5.805$$

3. Consider a European call option on a stock with strike price of \$50 and time to expiration of 1 year. An otherwise identical knock-in and knock-out call options with a barrier of \$57 trade for \$10.35 and \$5.15 respectively. Find the price of the standard call option.
Solution: Since we know that the standard call option is equal to the sum of knock-in and knock-out call options, therefore the price of the standard call option is $10.35 + 5.15 = 15.5$.

4. Assume that the Black-Scholes framework holds. A gap call option on a stock has a trigger price of \$115, a strike price of \$100, and a time to expiration of 2 years. The stock currently trades for \$105 per share and pays dividends with a continuously compounded annual yield of 0.05. The annual continuously compounded risk-free interest rate is 8%, and the relevant price volatility for the Black-Scholes formula is 0.3. Find the Black-Scholes price of this gap call.

Solution: We assume $K_1 = 100$ and $K_2 = 115$.

$$d_1 = \frac{ln(\frac{S_0}{K_2}) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = 0.13913$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.28513$$

$$C_{gap} = S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2) = 19.71$$

5. A stock pays dividends continuously at a rate proportional to its price. Consider a European gap option with expiration date T. If the stock price $S(t)$ at time T is greater than \$100, the payoff is $S(T) - 90$. Otherwise, the payoff is zero. You are given: (i) $S(0) = \$80$; (ii) The price of a European call option with expiration date T and strike price \$100 is \$4; (iii) The delta of the call option in (ii) is 0.2. Calculate the price of the gap option.

Solution: We assume $K_1 = 90$ and $K_2 = 100$.

$$C_{100} = S_0 e^{-\delta T} N(d_1) - K_2 e^{-rT} N(d_2) = 4$$

$$\Delta_{C_{100}} = e^{-\delta T} N(d_1) = 0.2$$

Therefore we can obtain $e^{-rT} N(d_2) = 0.12$. Then

$$C_{gap} = S_0 e^{-\delta T} N(d_1) - K_1 e^{-rT} N(d_2) = 5.2$$

6. An exchange call option with expiration of one year allows the owner to acquire one share of a stock A for one share of a stock B. The price of the option is \$2.16. Stock A pays dividends at the continuously compounded yield of 7%. Stock B pays no dividends. Stock A currently trades for \$50 and stock B trades for \$55. Find the value of an exchange option that allows the owner to give up one share of stock A for one share of stock B.

Solution: From the question, we are given the price of the exchange call option is 2.16. Then we need to calculate the price of the exchange put option. By put-call parity, we have

$$C - P = S_0 e^{-\delta_S T} - K_0$$

Therefore $P = 10.54$.

7. The risk-neutral price process of dividend paying stock is

$$\frac{dS(t)}{S(t)} = 0.06dt + 0.12d\tilde{Z}(t).$$

The continuously compounded yield is 0.05. It is known that the expected rate of return on the stock is twice the risk-free interest rate. Find α.

Solution: $r - \delta = 0.06$ and $\alpha = 2r$. Therefore $\alpha = 0.22$

8. Stock XYZ has the following 10 monthly closing prices on the last day of each calendar month in the first ten months of 2016 (the prices are listed in chronological order, from January 31, 2016 through October 31, 2016):

$$85, 93, 89, 82, 74, 80, 86, 87, 91, 89$$

For each of the ten-month options on XYZ stock listed in the problems below, determine the payoff of the option on October 31, 2016. Assume each option began on January 1, 2016, and expires on October 31, 2016, and that it can only be exercised on the expiration date.

(a) Plain-vanilla (ordinary) 90-strike European put.

(b) Plain-vanilla (ordinary) 80-strike European call.

(c) 90-strike look-back put option.
(d) 80-strike look-back call option.
(e) Geometric average strike call.
(f) 80-strike Arithmetic average price call.
(g) Down-and-out arithmetic-average-strike 75-barrier call option.
(h) Up-and-in 90-strike 90-barrier put option.

Solution: (a)Payoff $= (K - S_T)_+ = 1$.
(b)Payoff $= (S_T - K)_+ = 9$.
(c)Payoff $= (K - minS_t)_+ = 16$.
(d)Payoff $= (maxS_t - K)_+ = 13$.
(e)$GA = 85.43$, Payoff $= (S_T - GA)_+ = 3.57$.
(f)$AA = 85.6$, Payoff $= (AA - K)_+ = 5.6$.
(g)Since $S_5 = 74 < 75$ then Payoff $= 0$.
(h)Since $S_2 = 93 > 90$ then Payoff $= (K - S_T)_+ = 1$.

9. A one-year Asset-Call (i.e., an asset-or-nothing call option) on a share of a non-dividend- paying stock with a current value of 82 has an exercise price of 83. The volatility of the stock is 0.40, and the continuously-compounded risk-free interest rate is 6%. Find the price of the Asset-Call.
Solution:

$$C = S_0 e^{-\delta T} N(d_1)$$

$$d_1 = \frac{ln(\frac{S_0}{K}) + (r - \delta + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = 0.3197$$

Therefore $C = 51.28$.

10. Use the inverse transform method to simulate random variables of the following distributions, provided $u = 0.8945$.

(a) Normal distribution with mean 50 and variance 144
(b) Exponential distribution with mean 80
(c) Distribution with cdf $F(x) = 1 - \exp\left(-\left(\frac{x}{100}\right)^2\right)$
(d) Poisson distribution with $\lambda = 3$.

Solution: (a)

$$P(x \le y) = P(z \le \frac{y - 50}{\sqrt{144}}) = 0.8945$$

Therefore $y = 50 + 12 * N^{-1}(0.8945) = 65.01$.
(b)

$$F(x) = 1 - e^{-\lambda x} = 0.8945$$

Here we know $\lambda = \frac{1}{80}$. Hence we can obtain $x = 179.92$.
(c)

$$F(x) = 1 - \exp\left(-\left(\frac{x}{100}\right)^2\right) = 0.8945$$

Hence we can obtain $x = 149.97$.
(d)

$$f(n) = \frac{\lambda^n e^{-\lambda}}{n!}$$

n	f(n)	F(n)
0	0.0498	0.0498
1	0.1494	0.1991
2	0.2240	0.4232
3	0.2240	0.6472
4	0.1680	0.8153
5	0.1008	0.9161

Since $F(4) < u < F(5)$, then x=5.

Additional Problems for Math 567 Students (max. points = 20)

11. We know that the price for a cash-or nothing call is given by $C = e^{-r(T-t)}XN(d_2)$ and cash-or-nothing put is $e^{-r(T-t)}XN(-d_2)$. Prove that

$$\Delta_C = e^{-r(T-t)}X\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{1}{S_t\sigma\sqrt{T-t}}, \quad \Delta_P = -e^{-r(T-t)}X\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{1}{S_t\sigma\sqrt{T-t}},$$

Solution:

$$\Delta_C = \frac{\partial C}{\partial S_t} = e^{-r(T-t)}X\frac{\partial N(d_2)}{\partial S_t}$$

$$N(d_2) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_2} e^{-\frac{x^2}{2}}dx$$

$$d_2 = d_1 - \sigma\sqrt{(T-t)} = \frac{ln(\frac{S_t}{K}) + (r - \delta - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

Therefore

$$\Delta_C = e^{-r(T-t)}X\frac{\partial d_2}{\partial S_t}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} = e^{-r(T-t)}X\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{1}{S_t\sigma\sqrt{T-t}}$$

$$\Delta_P = \frac{\partial P}{\partial S_t} = e^{-r(T-t)}X\frac{\partial N(-d_2)}{\partial S_t} = -e^{-r(T-t)}X\frac{\partial d_2}{\partial S_t}\frac{1}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} = -e^{-r(T-t)}X\frac{e^{-d_2^2/2}}{\sqrt{2\pi}}\frac{1}{S_t\sigma\sqrt{T-t}}$$

12. Assume the Black-Scholes framework. Consider two non-dividend-paying stocks whose time t prices are denoted by $S_1(t)$ and $S_2(t)$, respectively. You are given:

(i) $S_1(0) = 10$ and $S_2(0) = 21$.
(ii) Stock 1's volatility is 0.25.
(iii) Stock 2's volatility is 0.15.
(iv) The correlation between the continuously compounded returns of the two stocks is -0.33.
(v) The continuously compounded risk-free interest rate is 7%.
(vi) A one-year European option with payoff $\max\{21 - \max\{2S_1(1), S_2(1)\}, 0\}$ has a current price of 0.29.

Consider a European option that gives its holder the right to buy either two shares of Stock 1 or one share of Stock 2 at a price of 21 one year from now. Calculate the current (time-0) price of this option.

[Hint]: Consider asset A with payoff function $\max\{2S_1(1), S_2(1)\}$, use put-call parity for call/put options on asset A.

Solution: Suppose that $X = \max\{2S_1(1), S_2(1)\}$ is the payoff of a certain asset A. Using (vi) we have the payoff of a European put option on asset A

$$\max\{21 - \max\{2S_1(1), S_2(1)\}, 0\} = \max\{21 - X, 0\} = 0.29$$

We are asked to find the time-0 price of a European call option with payoff, one year from today, given by

$$\max\{\max\{2S_1(1), S_2(1)\} - 21, 0\} = \max\{X - 21, 0\}$$

Using the put-call parity

$$C - P = S - Ke^{-rT}$$

we find $C = 0.29 + A_0 - 21e^{-0.07*1}$, where A_0 is the time-0 price of asset A. It follows that in order to find C we must find the value of A_0. Since

$$\max\{2S_1(1), S_2(1)\} = 2S_1(1) + \max\{S_2(1) - 2S_1(1), 0\}$$

the time-0 value of A is twice the time-0 value of stock 1 plus the time-0 value of an exchange call option that allows the owner to give two shares of stock 1 (strike asset) for one share of stock 2 (underlying asset). Hence,

$$A_0 = 2S_1(0) + ExchangeCallPrice$$

We next find the time-0 value of the exchange call option. We have

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} = 0.3313$$

$$d_1 = \frac{ln(\frac{S_2(0)}{2S_1(0)}) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} = 0.31292$$

$$d_2 = d_1 - \sigma\sqrt{T} = -0.01838$$

$$N(d_1) = 0.62283$$

$$N(d_2) = 0.49267$$

Therefore the value of the exchange call option is

$$S_2(0)N(d_1) - 2S_1(0)N(d_2) = 3.22603$$

Hence

$$A_0 = 20 + 3.22603 = 23.22603$$

and

$$C = 0.29 + 23.22603 - 21e^{-0.07*1} = 3.936$$