Brownian Motion and Diffusion Processes

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Background Information

- Brownian motion is the random motion of particles suspended in a fluid (a liquid or a gas) resulting from their collision with the quickly moving atoms or molecules in the fluid.
- This transport phenomenon is named after the botanist Robert Brown.
- The mathematical model of Brownian motion has numerous real-world applications, such as stock market analysis.

Definition

1-D Brownian Motion

- $B(0) = 0$
- $B_t \sim \mathscr{N}(0,t)$, $t \in \mathscr{R}_{\geq 0}$
- $B(t+\Delta t) - B(t)$ is independent of $B(s)$ for all $s \leq t$;

2-D Brownian Motion

- Created based on 1-D Brownian Motion
- $B(t,s) = (B_t, B_s)$
- $B_t \sim \mathscr{N}(0,t)$, $t \in \mathscr{R}_{\geq 0}$
- $B_s \sim \mathscr{N}(0,s)$, $s \in \mathscr{R}_{\geq 0}$

Kolmogorov Process

A **Kolmogorov process** is a collection of random variables $\{K_t | t \in \mathscr{R}_{\geq 0}\}$ such that $K_t = (B_t, \int_0^t B_s ds)$ where B_t is a one-dimensional Brownian motion.

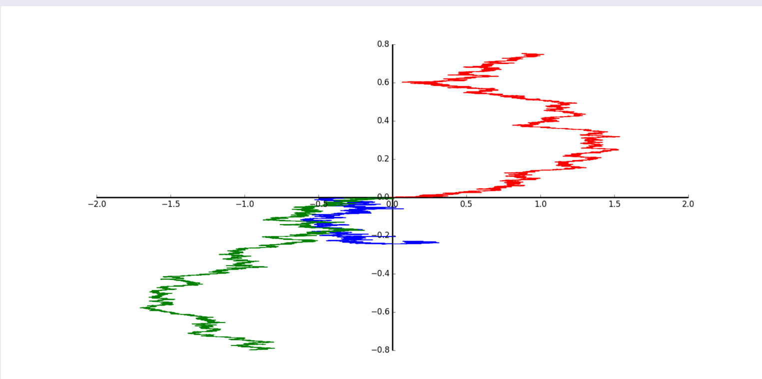


Figure: Kolmogorov process sample paths

Roto Transition

- $\{(V, \theta_1)\} \mapsto \{(X, Y, \theta_2)\}$
- $V \in \mathscr{R}, \theta_1 \in (-\pi, -\pi), (X, Y) \in \mathscr{R}^N, \theta_2 \in (-\pi, -\pi)$
- Give initial state, could find a function whose parameter is velocity and direction, which represent the process from initial state to ending state at a specific location and angel.
- Application: parallel parking

Brownian sheet

- A 1-D Brownian Sheet is a 2-Parameter, centered Gaussian process $B = \{B(s,t); s,t \geq 0\}$
- $\mathscr{E}\{B(s,t)B(s',t')\} \;=\; min(s,s') \;\times min(t,t')$

Our Work

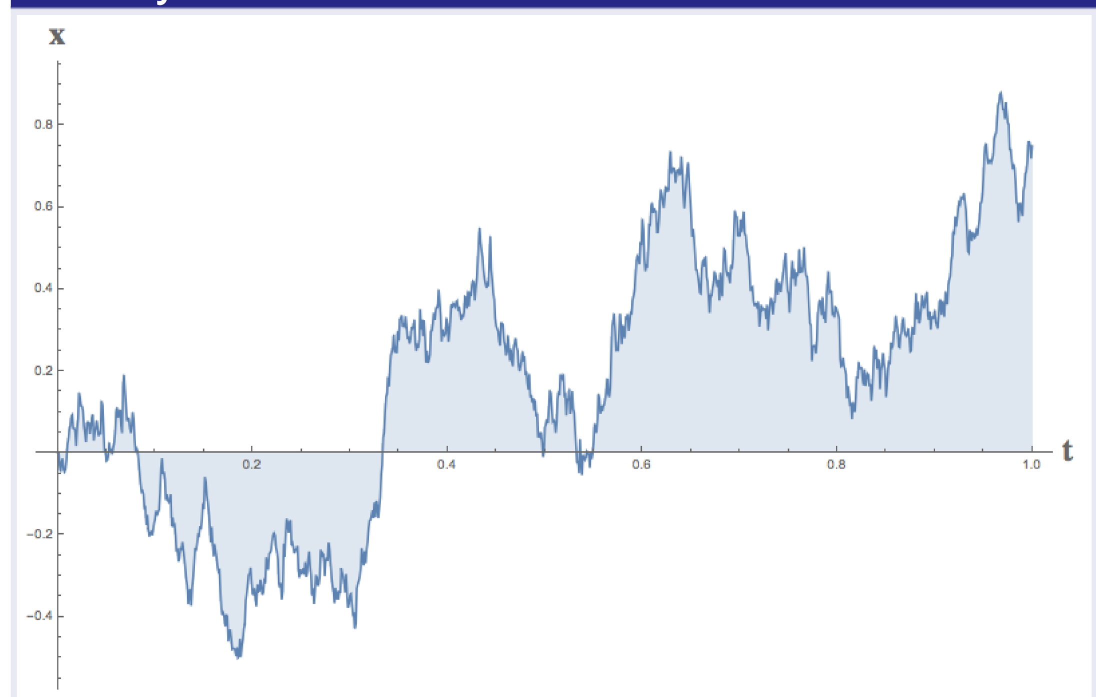
Simulation

We began by creating simulations of Brownian motion using Python and Matlab. Our simulations of Brownian motion are quite simple: we start a particle at the origin, and after each increment of time, we sample a normal distribution of $mean = 0$ and $variance = \frac{1}{\sqrt{\text{total steps}}}$, and add it to the previous location of the particle. Every step is independent, and we see a true Brownian motion from the simulation. We often chose to have 1000 steps per simulation, as it is big enough to see its random nature, but small enough to be computed efficiently.

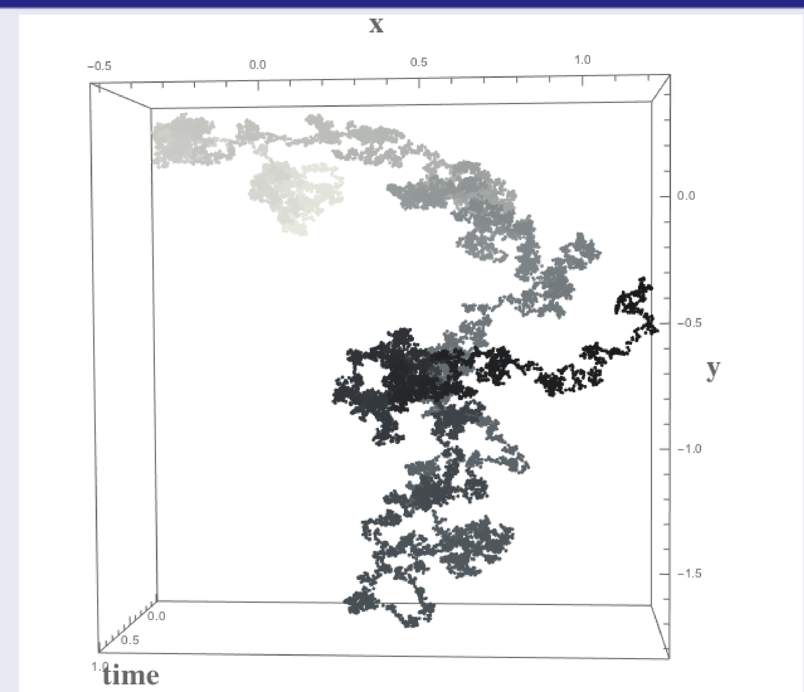
Extended Use of Simulation

We can extend these methods to simulate other topics. We can create 2-D Brownian motion by having two separate and independent simulations of Brownian Motion on each axis at the same time. We can observe the Kolmogorov process as well, where one dimension is standard Brownian motion, while the other dimension records the integral of that Brownian motion. We can extend our code further to explore roto-translational motion, where a Brownian motion determines a particle's movement forward or backward, while another random process allows it to turn left or right. This process models the possible motion of a car.

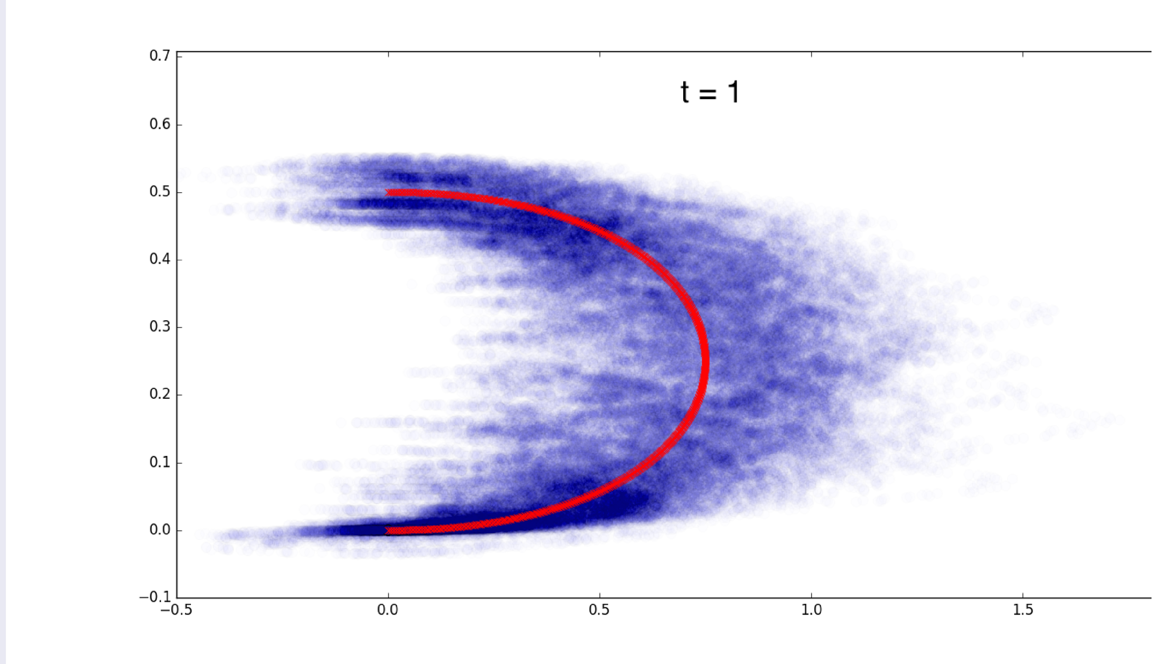
Gallery



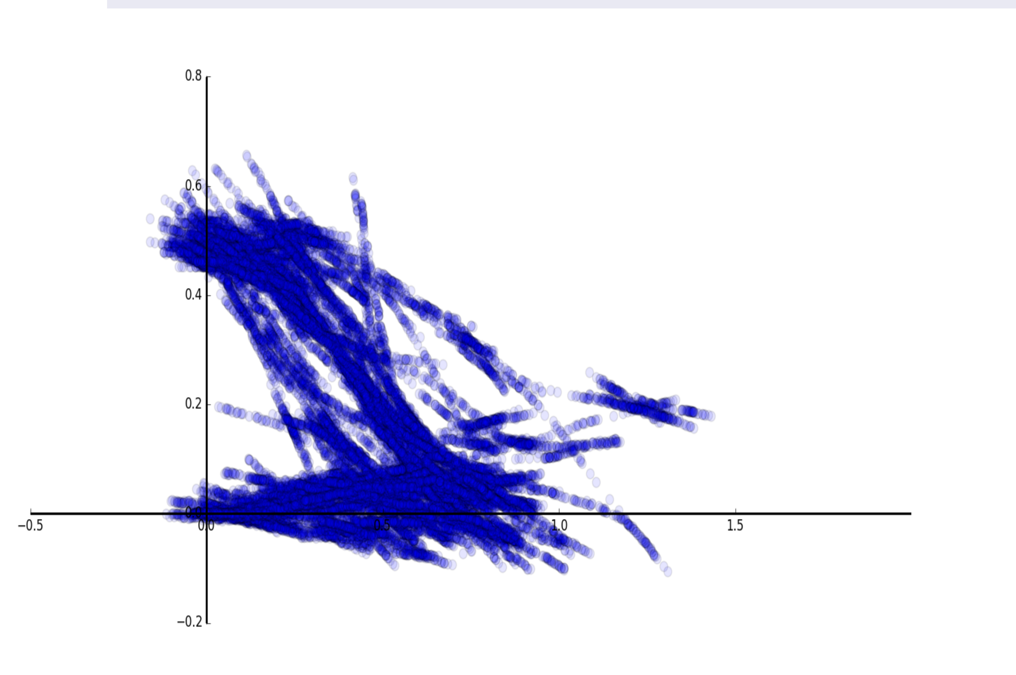
(a) 1-D Brownian Motion



(b) 2-D Brownian Motion



(c) Many sample paths of Kolmogorov process (blue) and an optimal path (red)



(d) Sample paths of roto-translation

Results and Further Work

- We have created simulations of 1-D and 2-D Brownian motions, the Kolmogorov process, and the diffusion process on the roto-translation group.
- After the simulations were completed, we turned our attention to the study of optimal paths in each of the diffusion processes mentioned above. Using an approach reminiscent of the Monte Carlo method, we ran our simulations multiple times, fixing the endpoints of our sample paths. By observing where the sample paths were concentrated, we could successfully infer the approximate form of optimal paths.
- In our study of optimal paths, we also observed phenomena associated with Hörmander's condition. That is, when we reduced the time variable in our simulations, we noticed that the Brownian motion and the diffusion process on the roto-translation group had a converging behavior, and the sample paths became more concentrated around the optimal path. For the Kolmogorov process, however, constraining the time variable resulted in an entirely new optimal path.
- At the moment, we are working to create simulations of other diffusion processes, such as the Brownian sheet and the diffusion process on the Heisenberg group. Moreover, we plan to review our code and find ways to improve the computational efficiency of our simulations, since at the moment they are very costly and can take hours to run.

References

- CAPOGNA, L., DANIELLI, D., PAULS, S., and TYSON, J. *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem.* Birkhäuser Basel, 2007.
- MÖRTERS, P., and PERES, Y. *Brownian Motion* [draft], 2008.

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