Conformal transforms and Doob's h-processes on Heisenberg groups

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Abstract

We study the stochastic processes that are images of Brownian motions on Heisenberg group $\mathbf{H}^{2n+1}$ under conformal maps. In particular, we obtain that Cayley transform maps Brownian paths in $\mathbf{H}^{2n+1}$ to a time changed Brownian motion on CR sphere $\mathbb{S}^{2n+1}$ conditioned to be at its south pole at a random time. We also obtain that the inversion of Brownian motion on $\mathbf{H}^{2n+1}$ started from $x \neq 0$, is up to time change, a Brownian bridge on $\mathbf{H}^{2n+1}$ conditioned to be at the origin.

Contents

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1 Introduction

The Brownian motions on sub-Riemannian model spaces has been widely studied in recent years. Due to strong symmetries of the model spaces, explicit computations analysis can be conducted (see [7], [4], [1], [2], and [3]). In this paper we focus on the relationships between Brownian motion on Heisenberg group and its images under certain conformal maps, namely Cayley transform and Kelvin transform.

Let $\mathbf{H}^{2n+1}$ be a $2n+1$ dimensional Heisenberg group that lives in $\mathbb{C}^n \times \mathbb{R}$ with coordinates $(z,t) = (z_1, \ldots, z_n, t)$ where $z_j = x_j + iy_j$. It has the group law

$$(z,t)(z',t') = (z+z', t+t' + \mathbf{Im} z\bar{z'}).$$

It is a flat model space of sub-Riemannian manifolds. There is a canonical sub-Laplacian on $\mathbf{H}^{2n+1}$:

$$\bar{L}_{\mathbf{H}^{2n+1}} = \sum_{j=1}^{n} \left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 2y_j \frac{\partial^2}{\partial x_j \partial t} - 2x_j \frac{\partial^2}{\partial y_j \partial t} + |z_j|^2 \frac{\partial^2}{\partial t^2} \right)$$

The Brownian motion on $\mathbf{H}^{2n+1}$ issued from $x' \in \mathbf{H}^{2n+1}$ is the strong Markov process that is generated by $\frac{1}{2}\bar{L}_{\mathbf{H}^{2n+1}}$.

Cayley transform is known to be a bi-holomorphic map between the Siegel domain Ω^{n+1} and a unit ball in $\mathbb{C}^{n+1}$. The restriction of Cayley transform on its boundary therefore provides a conformal map between $\mathbf{H}^{2n+1}$ and the unit sphere $\mathbb{S}^{2n+1}$ in $\mathbb{C}^{n+1}$. If we consider the image of a Brownian path on $\mathbf{H}^{2n+1}$ under Cayley transform, it then turns out to be a $\mathbb{S}^{2n+1}$-valued process. In particular, it is a time changed version of a Brownian path on $\mathbb{S}^{2n+1}$ conditioned to be at the south pole at a random time. Below we state our main result.

Theorem 1.1 *The Brownian motion on $\mathbf{H}^{2n+1}$ issued from x' is mapped by Cayley transform $\mathcal{C}_1$ to a time-changed Brownian motion on $\mathbb{S}^{2n+1}$ issued from $x = \mathcal{C}_1(x')$ and conditioned to be at the south pole $-e_n$ at time T, where T is an independent random variable with distribution*

$$\mathbb{P}_x^h [T > t] = \frac{\int_t^{+\infty} e^{-n^2 s} p_s(-e_n, x) ds}{\int_0^{+\infty} e^{-n^2 t} p_t(-e_n, x) dt}. \tag{1.1}$$

Here $p_t(x,y)$ denotes the subelliptic heat kernel on $\mathbb{S}^{2n+1}$.

This result extends the result by Carne in [5], where he proved that the Stereographic projection from $\mathbb{R}^n$ to S^n maps Brownian paths in $\mathbb{R}^n$ to the paths of conditioned Brownian motion on S^n.

Another object of our study is to probabilistically interpret the relation between the Brownian motion on $\mathbf{H}^{2n+1}$ started from any $x' \neq 0$ and its image under the inversion map, namely the Kelvin transform. This type of question was first posed by Schwartz (see [10]), who asked how Brownian motion in $\mathbb{R}^n$ can be interpreted as a Brownian bridge conditioned to be at the "ideal point at infinity". A probabilistic approach was provided by

Yor in [11]. In the present paper, we obtain the result in a setting of a flat sub-Riemannian manifold. The inversion of Brownian motion on $\mathbf{H}^{2n+1}$ issued from $x \neq 0$ turns out to be a Brownian bridge conditioned to be at the origin up to time change.

Theorem 1.2 *The Brownian motion on $\mathbf{H}^{2n+1}$ generated by $\frac{1}{2}L_{\mathbf{H}^{2n+1}}$ and issued from $x' \neq 0$ is mapped by Kelvin transform to a time-changed $\mathbf{H}^{2n+1}$-valued Brownian motion conditioned to be at the origin at $t = \infty$.*

The approaches to both results follow the idea of Carne. By analyzing the radial part of the corresponding conformal sub-Laplacians on $\mathbb{S}^{2n+1}$ and on $\mathbf{H}^{2n+1}$, we are able to obtain the relationship between Markov processes that are generated by $\frac{1}{2}L_{\mathbb{S}^{2n+1}}$ and $\frac{1}{2}L_{\mathbf{H}^{2n+1}}$ respectively through an argument of Doob's h-processes.

In the next section, we deduce Theorem 1.1 after a detailed discussion of Cayley transform and radial process or Brownian motions on $\mathbb{S}^{2n+1}$ and $\mathbf{H}^{2n+1}$. In section 3 we focus on the inverse transform on $\mathbf{H}^{2n+1}$ and the proof of Theorem 1.2.

2 Cayley transformation and Doob's h-process

2.1 Cayley transform on CR model spaces

Cayley transforms on CR model spaces are natural analogues of stereographic projections on Riemannian models. Let $B^{n+1} = \{\zeta \in \mathbb{C}^{n+1} : |\zeta| < 1\}$ be the unit ball in $\mathbb{C}^{n+1}$ and $\Omega^{2n+1} = \{(z, w) \in \mathbb{C}^n \times \mathbb{C}, \mathbf{Im}(w) > |z|^2\}$ the Siegel domain. The Cayley transform $\mathcal{C} : B^{2n+1} \to \Omega^{n+1}$ is a biholomorphic map such that (see [6])

$$\mathcal{C} : (\zeta_1, \dots, \zeta_{n+1}) \to \left(\frac{\zeta_1}{1+\zeta_{n+1}}, \dots, \frac{\zeta_n}{1+\zeta_{n+1}}, i\frac{1-\zeta_{n+1}}{1+\zeta_{n+1}} \right), \quad \zeta^{n+1} \neq -1.$$

Let $\mathbb{S}^{2n+1} = \{\zeta \in \mathbb{C}^{n+1}, |\zeta| = 1\}$ be the unit sphere in $\mathbb{C}^{n+1}$. It also appears as a model space of CR manifolds. The restriction of $\mathcal{C}$ to the CR sphere $\mathbb{S}^{2n+1}$ minus a point gives a CR diffeomorphism to the boundary of the Siegel domain $\partial\Omega^{2n+1}$, which may be identified with the Heisenberg group $\mathbf{H}^{2n+1}$ through the CR isomorphism $\varphi : \mathbf{H}^{2n+1} \to \partial\Omega^{2n+1}$. For any $(z, t) \in \mathbf{H}^{2n+1}$,

$$\varphi(z, t) = (z, 2t + i|z|^2). \tag{2.2}$$

We denote the north pole of $\mathbb{S}^{2n+1}$ by $e_n = \{0, \dots, 0, 1\}$ and corresponding the south pole by $-e_n$. Now we consider the CR equivalence between Heisenberg group and CR sphere minus the south pole $\mathcal{C}_1 : \mathbf{H}^{2n+1} \to \mathbb{S}^{2n+1} \setminus \{-e_n\}$. It is then given by $\mathcal{C}_1 = \mathcal{C}^{-1} \circ \varphi$. In local coordinates we have for any $(z, t) = (z_1, \dots, z_n, t) \in \mathbf{H}^{2n+1}$,

$$\mathcal{C}_1 : (z, t) \to \left(\frac{2z_1}{(1+|z|^2) - 2it}, \dots, \frac{2z_n}{(1+|z|^2) - 2it}, \frac{1-|z|^2+2it}{1+|z|^2-2it} \right). \tag{2.3}$$

It is a conformal map with inverse $\mathcal{C}_1^{-1} : \mathbb{S}^{2n+1} \setminus \{-e_n\} \to \mathbf{H}^{2n+1}$,

$$\mathcal{C}_1^{-1} : (\zeta_1, \cdots \zeta_{n+1}) \to \left(\frac{\zeta_1}{1+\zeta_{n+1}}, \dots, \frac{\zeta_n}{1+\zeta_{n+1}}, \frac{i}{2}\frac{\overline{\zeta_{n+1}} - \zeta_{n+1}}{|1+\zeta_{n+1}|^2} \right). \tag{2.4}$$

Since $\mathbb{S}^{2n+1}$ is a model space of sub-Riemannian manifold with the Hopf fibration $\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{CP}^n$, it is more convenient for us to use the so-called cylindrical coordinates that carries the structural information and are given by

$$(w, \theta) \to \frac{e^{i\theta}}{\sqrt{1+|w|^2}}\,(w, 1)\,,$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and $w = \zeta/\zeta_{n+1} \in \mathbb{CP}^n$. Here $w = (w_1, \cdots, w_n)$ parametrizes the complex lines passing through the origin, and θ determines a point on the line that is of unit distance from the north pole. Let $|w| = \tan r_S$, $r_S \in [0, \pi/2)$, then we have $\mathcal{C}_1^{-1}$ in cylindrical coordinates given by

$$\mathcal{C}_1^{-1} : \left(\frac{e^{i\theta}}{\sqrt{1+|w|^2}}\,(w,1)\right) \to \left(\frac{e^{i\theta}\cos r_S + \cos^2 r_S}{1+\cos^2 r_S + 2\cos r_S \cos\theta} w, \frac{\cos r_S \sin\theta}{1+\cos^2 r_S + 2\cos r_S\cos\theta}\right).$$

Let $\psi_S : \mathbb{S}^{2n+1} \to [0, \pi/2) \times \mathbb{R}/2\pi\mathbb{Z}$ be such that

$$\psi_S\left(\frac{e^{i\theta}}{\sqrt{1+|w|^2}}\,(w,1)\right) = (r_S, \theta)$$

and $\psi_H : \mathbf{H}^{2n+1} \to \mathbb{R}_{\geq 0} \times \mathbb{R}$ be such that

$$\psi_H\,(z,t) = (r_H, t),$$

where $r_H = \sqrt{\sum_{j=1}^n |z_j|^2}$. We define a map $\mathbb{R}_{\geq 0} \times \mathbb{R} \to [0, \pi/2) \times \mathbb{R}/2\pi\mathbb{Z}$ by the chart below, and by abusing of notation we denote it by $\mathcal{C}_1$:

$$\begin{array}{ccc} \mathbf{H}^{2n+1} & \xrightarrow{\mathcal{C}_1} & \mathbb{S}^{2n+1} \\ \Big\downarrow \psi_H & & \Big\downarrow \psi_S \\ \mathbb{R}_{\geq 0} \times \mathbb{R} & \xrightarrow{\mathcal{C}_1} & [0, \pi/2) \times \mathbb{R}/2\pi\mathbb{Z} \end{array}$$

We easily compute that

$$\mathcal{C}_1 : (r_H, t) \to \left(\arcsin\left(\frac{2r_H}{\sqrt{(1+r_H^2)^2+4t^2}}\right), \arcsin\left(\frac{4t}{\sqrt{(1+r_H^2)^2+4t^2}\sqrt{(1-r_H^2)^2+4t^2}}\right)\right)$$

and

$$\mathcal{C}_1^{-1} : (r_S, \theta) \to \left(\frac{\sin r_S}{\sqrt{1+\cos^2 r_S + 2\cos r_S\cos\theta}}, \frac{\cos r_S \sin\theta}{1+\cos^2 r_S + 2\cos r_S \cos\theta}\right).$$

2.2 Brownian motion and Doob's h-process

Now we consider the Markov processes that are generated by sub-Laplacians $\bar{L}_{\mathbf{H}^{2n+1}}$ and $\bar{L}_{\mathbb{S}^{2n+1}}$, which are referred to as Brownian motions on $\mathbf{H}^{2n+1}$ and $\mathbb{S}^{2n+1}$ respectively throughout this paper. Due to the radial symmetries of these diffusion processes, it is sufficient for us to consider only the radial part of the sub-Laplacians.

We denote by $L_{\mathbf{H}^{2n+1}}$ the radial part of the sub-Laplacian on $\mathbf{H}^{2n+1}$ in coordinates $(r_H,\, t)$, it is defined on the space $D_H = \{f \in C^\infty(\mathbb{R}_{\geq 0} \times \mathbb{R}, \mathbb{R}), \frac{\partial f}{\partial r_H}|_{r_H=0} = 0\}$. Let $L_{\mathbb{S}^{2n+1}}$ be the radial part of $\bar{L}_{\mathbb{S}^{2n+1}}$ in cylindric coordinates $(r_S,\, \theta)$, with domain $D_S = \{f \in C^\infty([0, \frac{\pi}{2}) \times \mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}), \frac{\partial f}{\partial r_S}|_{r_S=0} = 0\}$. Then for any $f \in D_H$ and $g \in D_S$, we have

$$\bar{L}_{\mathbf{H}^{2n+1}}(f \circ \psi_H) = (L_{\mathbf{H}^{2n+1}} f) \circ \psi_H, \quad \bar{L}_{\mathbb{S}^{2n+1}}(g \circ \psi_S) = (L_{\mathbb{S}^{2n+1}} g) \circ \psi_S.$$

It is known that $L_{\mathbb{S}^{2n+1}}$ is essentially self-adjoint with respect to the volume measure $d\mu_{\mathbb{S}^{2n+1}} = \frac{2\pi^n}{\Gamma(n)}(\sin r_S)^{2n-1} \cos r_S dr_S d\theta$ on $\mathbb{S}^{2n+1}$, and $L_{\mathbf{H}^{2n+1}}$ is essentially self-adjoint with respect to the volume measure $d\mu_{\mathbf{H}^{2n+1}} = \frac{2\pi^n}{\Gamma(n)} r_H^{2n-1} dr_H dt$ on $\mathbf{H}^{2n+1}$. Moreover, we have explicitly

$$L_{\mathbf{H}^{2n+1}} = \frac{\partial^2}{\partial r_H^2} + \frac{2n-1}{r_H}\frac{\partial}{\partial r_H} + r_H^2 \frac{\partial^2}{\partial t^2} \tag{2.5}$$

and (see [2], [1], also [8])

$$L_{\mathbb{S}^{2n+1}} = \frac{\partial^2}{\partial r_S^2} + ((2n-1)\cot r_S - \tan r_S)\frac{\partial}{\partial r_S} + \tan^2 r_S \frac{\partial^2}{\partial \theta^2}. \tag{2.6}$$

Let us consider Green function of the conformal sub-Laplacian $-L_{\mathbb{S}^{2n+1}} + n^2$ with pole $(0,0)$ (the north pole of $\mathbb{S}^{2n+1}$) and denote it by $G_{\mathbb{S}^{2n+1}}$. From [2] we have

$$G_{\mathbb{S}^{2n+1}}((0,0),(r_S,\theta)) = \frac{\Gamma\left(\frac{n}{2}\right)^2}{8\pi^{n+1}(1 - 2\cos r_S \cos\theta + \cos^2 r_S)^{n/2}}. \tag{2.7}$$

On the other hand the Green function of $-L_{\mathbf{H}^{2n+1}}$ with respect to $d\mu_{\mathbf{H}^{2n+1}}$ is given by

$$G_{\mathbf{H}^{2n+1}}((0,0),(r_H,t)) = \frac{\Gamma\left(\frac{n}{2}\right)^2}{8\pi^{n+1}(r_H^4 + 4t^2)^{n/2}} \tag{2.8}$$

We consider $h \in D_S$, such that for any $(r_S, \theta) \in [0, \frac{\pi}{2}) \times \mathbb{R}/2\pi\mathbb{Z}$,

$$h(r_S, \theta) = 1 + 2\cos r_S \cos\theta + \cos^2 r_S, \tag{2.9}$$

and $H \in D_H$, such that for any $(r_H, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$,

$$H(r_H, t) = \frac{4}{(1 + r_H^2)^2 + 4t^2}. \tag{2.10}$$

It is an easy fact that h and H are harmonic functions with poles $(0,\pi)$ and $(0,0)$ respectively. Moreover, we have

$$H = \mathcal{C}_1^* h = h \circ \mathcal{C}_1.$$

From (2.7) and (2.8) we can easily observe that

$$G_{\mathbb{S}^{2n+1}}((0,0),(r_S,\theta))(1+2\cos r_S\cos\theta+\cos^2 r_S)^{\frac{n}{2}}=(\mathcal{C}_1^{-1*}G_{\mathbf{H}^{2n+1}})((0,0),(r_S,\theta)).$$

In fact, for any $x,y\in[0,\frac{\pi}{2})\times\mathbb{R}/2\pi\mathbb{Z}$ we have

$$G_{\mathbb{S}^{2n+1}}(x,y)=(\mathcal{C}_1^{-1*}G_{\mathbf{H}^{2n+1}})(x,y)h(x)^{-\frac{n}{2}}h(y)^{-\frac{n}{2}}. \tag{2.11}$$

From this we can then deduce the relation between $L_{\mathbf{H}^{2n+1}}$ and $L_{\mathbb{S}^{2n+1}}-n^2$.

Theorem 2.1 *For any function $f\in D_S$, the relation of $L_{\mathbf{H}^{2n+1}}$ and $L_{\mathbb{S}^{2n+1}}-n^2$ via Cayley transform is given by*

$$h^{(\frac{n}{2}+1)}(-L_{\mathbb{S}^{2n+1}}+n^2)\left(h^{-\frac{n}{2}}f\right)=-(\mathcal{C}_{1*}L_{\mathbf{H}^{2n+1}})f \tag{2.12}$$

where h is as in (2.9).

Proof. For any $f\in D_S$, let $F\in D_H$ be such that $F=(\mathcal{C}_1)^*f=f\circ\mathcal{C}_1$. We assume for some $\sigma_1,\sigma_2\in D_S$ it holds that for any $x\in[0,\frac{\pi}{2})\times\mathbb{R}/2\pi\mathbb{Z}$,

$$(-L_{\mathbb{S}^{2n+1}}+n^2)\,(\sigma_1 f)\,|_x=-\sigma_2\,(L_{\mathbf{H}^{2n+1}})\,(\mathcal{C}_1^*f)|_{\mathcal{C}_1^{-1}(x)}.$$

It then amounts to find σ_1,σ_2. Let $g=-L_{\mathbf{H}^{2n+1}}F$, then $F=(-L_{\mathbf{H}^{2n+1}})^{-1}g$. The above equation is equivalent to

$$\sigma_1\cdot\left((-L_{\mathbf{H}^{2n+1}})^{-1}g\right)\circ\mathcal{C}_1^{-1}=(-L_{\mathbb{S}^{2n+1}}+n^2)^{-1}(\sigma_2(g\circ\mathcal{C}_1^{-1})). \tag{2.13}$$

Therefore, for all $x\in[0,\frac{\pi}{2})\times\mathbb{R}/2\pi\mathbb{Z}$, we have

$$\int G_{\mathbf{H}^{2n+1}}(\mathcal{C}_1^{-1}(x),v)g(v)d\mu_{\mathbf{H}^{2n+1}}v=\sigma_1^{-1}(x)\int G_{\mathbb{S}^{2n+1}}(x,y)\sigma_2(y)g(\mathcal{C}_1^{-1}(y))d\mu_{\mathbb{S}^{2n+1}}y \tag{2.14}$$

where $G_{\mathbb{S}^{2n+1}}$ and $G_{\mathbf{H}^{2n+1}}$ are Green functions as in (2.7) and (2.8). Moreover by changing variable $y=\mathcal{C}_1(v)$, the right hand side of the above equation writes

$$\sigma_1^{-1}(x)\int G_{\mathbb{S}^{2n+1}}(x,\mathcal{C}_1(v))\sigma_2(\mathcal{C}_1(v))g(v)|J_{\mathcal{C}_1}(v)|d\mu_{\mathbf{H}^{2n+1}}v, \tag{2.15}$$

where $|J_{\mathcal{C}_1}(v)|$ is the Jacobi determinant. We can easily compute that

$$|J_{\mathcal{C}_1}(v)|=H^{n+1}(v),$$

where H is given as in (2.10). Therefore (2.15) becomes

$$\sigma_1^{-1}(x)\int G_{\mathbb{S}^{2n+1}}(x,\mathcal{C}_1(v))\sigma_2(\mathcal{C}_1(v))g(v)H^{n+1}(v)d\mu_{\mathbf{H}^{2n+1}}v.$$

By plugging in (2.11) and comparing to (2.14), we obtain for all $x,y\in\mathbb{S}^{2n+1}$

$$\begin{cases}\sigma_1(x)=h^{-\frac{n}{2}}(x)\\ \sigma_2(y)=h^{-(1+\frac{n}{2})}(y),\end{cases}$$

hence the conclusion. □

Corollary 2.2 *For any function $f \in D_S$, we have that*

$$(\mathcal{C}_{1*}L_{\mathbf{H}^{2n+1}})f = h\left(L_{\mathbb{S}^{2n+1}}f + \frac{2\Gamma_{\mathbb{S}^{2n+1}}(h^{-\frac{n}{2}}, f)}{h^{-\frac{n}{2}}}\right) \tag{2.16}$$

where $\Gamma_{\mathbb{S}^{2n+1}}(f,g) = \frac{1}{2}(L_{\mathbb{S}^{2n+1}}(fg) - fL_{\mathbb{S}^{2n+1}}g - gL_{\mathbb{S}^{2n+1}}f)$ *for any* $f, g \in D_S$.

Proof. Notice that

$$(L_{\mathbb{S}^{2n+1}} - n^2)(h^{-\frac{n}{2}}) = 0.$$

hence

$$h^{\frac{n}{2}}(L_{\mathbb{S}^{2n+1}} - n^2)(h^{-\frac{n}{2}}f) = L_{\mathbb{S}^{2n+1}}f + 2h^{\frac{n}{2}}\Gamma_{\mathbb{S}^{2n+1}}(h^{-\frac{n}{2}}, f).$$

□

Now we are ready to prove the main result.

Proof of Theorem 1.1 The proof follows two steps.

Step 1: Notice that $h^{-\frac{n}{2}}$ is the Green function of the conformal sub-Laplacian $L_{\mathbb{S}^{2n+1}} - n^2$ with pole $(\pi/2, 0)$ (the south pole $-e_n$ of $\mathbb{S}^{2n+1}$). For any $f \in D_S$ we let

$$L^h f := L_{\mathbb{S}^{2n+1}}f + \frac{2\Gamma_{\mathbb{S}^{2n+1}}(h^{-\frac{n}{2}}, f)}{h^{-\frac{n}{2}}} = \frac{L_{\mathbb{S}^{2n+1}}(h^{-\frac{n}{2}}f)}{h^{-\frac{n}{2}}} - n^2 f. \tag{2.17}$$

Let X_t^h and X_t be Markov processes generated by $\frac{1}{2}L^h$ and $\frac{1}{2}L_{\mathbb{S}^{2n+1}}$, issued from $x \in \mathbb{S}^{2n+1}$. We first prove that X_t^h is X_t conditioned to be at the south pole $-e_n$ at time T, where T is a random time with distribution (1.1).

It is sufficient to prove that for any $f \in D_S$,

$$\mathbb{E}_x\left[f(X_t^h)\right] = \mathbb{E}_x\left[f(X_t)\mathbb{1}_{t<T} | X_T = -e_n\right] \tag{2.18}$$

Let P_t^h and P_t be the heat semigroups generated by L^h and $L_{\mathbb{S}^{2n+1}}$ respectively, then by iterating (2.17) it is not hard to obtain for any $x \in \mathbb{S}^{2n+1}$,

$$P_t^h(f(x)) = h(x)^{\frac{n}{2}}e^{-tn^2}P_t(h^{-\frac{n}{2}}(x)f(x)),$$

that is

$$\mathbb{E}_x\left[f(X_t^h)\right] = \frac{1}{h^{-\frac{n}{2}}(x)}e^{-tn^2}\mathbb{E}_x\left[h^{-\frac{n}{2}}(X_t)f(X_t)\right] = \mathbb{E}_x\left[\frac{e^{-tn^2}h^{-\frac{n}{2}}(X_t)}{h^{-\frac{n}{2}}(x)}f(X_t)\right].$$

Proving (2.18) is then equivalent to proving

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{t<T} | X_T = -e_n\right] = \mathbb{E}_x\left[\frac{e^{-tn^2}h^{-\frac{n}{2}}(X_t)}{h^{-\frac{n}{2}}(x)}f(X_t)\right]. \tag{2.19}$$

Note that

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{t<T}|X_T=-e_n\right]=\frac{\mathbb{E}_x\left[f(X_t)\mathbb{1}_{t<T}\mathbb{1}_{X_T=-e_n}\right]}{\mathbb{E}_x\left[X_T=-e_n\right]}.$$

Assume T is an exponential random variable with parameter $-n^2$ under the original probability measure, we have

$$\mathbb{E}_x\left[f(X_t)\mathbb{1}_{t<T}\mathbb{1}_{X_T=-e_n}\right]=\mathbb{E}_x\left[e^{-tn^2}h^{-\frac{n}{2}}(X_t)f(X_t)\right]$$

and

$$\mathbb{E}_x\left[X_T=-e_n\right]=\int_0^{+\infty}p_t(x,-e_n)e^{-n^2t}dt=h^{-\frac{n}{2}}(x).$$

Thus (2.19) holds when T is an exponential random variable under the original probability measure. Switching to the conditioned probability measure, T then has the distribution

$$\mathbb{P}_x^h\left[T>t\right]=e^{-n^2t}\frac{\mathbb{E}_x\left[h^{-\frac{n}{2}}(X_t)\right]}{h^{-\frac{n}{2}}(x)}=\frac{\int_t^{+\infty}e^{-n^2s}p_s(-e_n,x)ds}{\int_0^{+\infty}e^{-n^2t}p_t(-e_n,x)dt}.$$

Step 2: Next we prove the time change. Let Y_t be the Markov process generated by $\frac{1}{2}L_{\mathbf{H}^{2n+1}}$ and issued from $\mathcal{C}_1^{-1}(x)$, we claim that Y_t is mapped by Cayley transform to a time-changed version of X^h, i.e.,

$$X^h_{\mathcal{A}_t}=\mathcal{C}_1(Y_t) \tag{2.20}$$

where the time change is given by $\mathcal{A}_t=\int_0^t H(Y_s)^{-1}ds$. To see this, we consider for any $F=f\circ\mathcal{C}_1\in D_H$, the associated martingale M_t^F that is given by

$$M_t^F=F(Y_t)-\frac{1}{2}\int_0^t L_{\mathbf{H}^{2n+1}}F(Y_s)ds.$$

By plugging in (2.20), (2.16) and (2.17) we have

$$M_t^F=f(X^h_{\mathcal{A}_t})-\frac{1}{2}\int_0^t(L_{\mathbf{H}^{2n+1}}F)\circ\mathcal{C}_1^{-1}(X^h_{\mathcal{A}_s})ds=f(X^h_{\mathcal{A}_t})-\frac{1}{2}\int_0^t H(Y_s)L^hf(X^h_{\mathcal{A}_s})ds.$$

Let σ_t be the hitting time such that $\sigma_t=\inf\{u,\mathcal{A}_u>t\}$, then clearly $\mathcal{A}_{\sigma_t}=t=\sigma_{\mathcal{A}_t}$. By changing variable $s=\sigma_u$ we obtain

$$M_t^F=f(X^h_{\mathcal{A}_t})-\frac{1}{2}\int_0^{\sigma_{\mathcal{A}_t}}H(Y_s)L^hf(X^h_{\mathcal{A}_s})ds=f(X^h_t)-\frac{1}{2}\int_0^{\mathcal{A}_t}H(Y_{\sigma_u})L^hf(X^h_u)\sigma'_udu.$$

Note for any $u>0$ we have $u=\mathcal{A}_{\sigma_u}=\int_0^{\sigma_u}H(Y_s)ds$. This implies that

$$1=H(Y_{\sigma_u})\sigma'_u.$$

Therefore

$$M_t^F=f(X^h_t)-\frac{1}{2}\int_0^{\mathcal{A}_t}L^hf(X^h_u)du,$$

and it completes the proof.

3 Inversion of Brownian motions on Heisenberg group

In this section we consider the inversion of Brownian motion on Heisenberg group. First we construct the inverse map by composing two Cayley transforms $\mathcal{C}_1$ and $\mathcal{C}_2$, between $\mathbf{H}^{2n+1}$ and $\mathbb{S}^{2n+1}$ minus a point ($-e_n$ and e_n respectively). We have already discussed $\mathcal{C}_1$ in the previous section. Now let us consider $\mathcal{C}_2 : \mathbf{H}^{2n+1} \to \mathbb{S}^{2n+1} \setminus \{e_n\}$ where e_n is the north pole on $\mathbb{S}^{2n+1}$. We have

$$\mathcal{C}_2 : (z,t) \to \left(\frac{2z_1}{1+|z|^2+2it}, \dots, \frac{2z_n}{1+|z|^2+2it}, -\frac{1-|z|^2-2it}{1+|z|^2+2it} \right).$$

and

$$\mathcal{C}_2^{-1} : \{\zeta_1, \cdots \zeta_{n+1}\} \to \left\{ \frac{\zeta_1}{1-\zeta_{n+1}}, \dots, \frac{i}{2} \frac{\overline{\zeta_{n+1}} - \zeta_{n+1}}{|1-\zeta_{n+1}|^2} \right\}.$$

Let $\mathcal{K} : \mathbf{H}^{2n+1} \setminus \{0\} \longrightarrow \mathbf{H}^{2n+1} \setminus \{0\}$ be such that $\mathcal{K} = \mathcal{C}_2^{-1} \circ \mathcal{C}_1$, then

$$\mathcal{K} : (z_1, \cdots z_n, t) \to \left(\frac{z_1}{|z|^2 - 2it}, \dots, \frac{z_n}{|z|^2 - 2it}, \frac{t}{|z|^4 + 4t^2} \right).$$

Clearly $\mathcal{K}$ is an involution on $\mathbf{H}^{2n+1} \setminus \{0\}$ and preserve the Korányi ball $\{(z,t) \in \mathbf{H}^{2n+1}, |z|^4 + 4t^2 = 1\}$. Indeed it is the Kelvin transform generalized to Heisenberg group (see [9]).

For any $(r_H, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$ and $(r_S, \theta) \in [0, \frac{\pi}{2}) \times \mathbb{R}/2\pi\mathbb{Z}$, we let $\tilde{h}(r_S, \theta) = 1 + \cos^2 r_S - 2\cos r_S \cos\theta$ and $\tilde{H}(r_H, t) = \frac{4(r_H^4 + 4t^2)}{(1+r_H^2)+4t^2}$, then $\mathcal{K}^* H = (\mathcal{C}_2 \circ \mathcal{C}_1^{-1})^* H = \tilde{H}$. Moreover, simple calculations show that

$$\tilde{h} = (\mathcal{C}_2^{-1})^* H, \quad h = (\mathcal{C}_2^{-1})^* \tilde{H}, \quad \tilde{h} = (\mathcal{C}_1^{-1})^* \tilde{H}.$$

Let $N(r_H, t) = r_H^4 + 4t^2$. By comparing the conformal Laplacians induced by $\mathcal{C}_1$ and $\mathcal{C}_2$, we obtain the following relation.

Theorem 3.1 *For any function $F \in D_H$,*

$$(\mathcal{K}_* L_{\mathbf{H}^{2n+1}})F = N^{\frac{n}{2}+1} L_{\mathbf{H}^{2n+1}}(N^{-n/2} F).$$

Proof. First we notice that for all $f \in D_S$,

$$\tilde{h}^{\frac{n}{2}+1}(-L_{\mathbb{S}^{2n+1}} + n^2)\left(\tilde{h}^{-\frac{n}{2}} f\right) = -(\mathcal{C}_{2*} L_{\mathbf{H}^{2n+1}}) f.$$

Together with (2.12) we obtain

$$h^{-(\frac{n}{2}+1)}(\mathcal{C}_{1*} L_{\mathbf{H}^{2n+1}})(h^{\frac{n}{2}} f) = \tilde{h}^{-(\frac{n}{2}+1)}(\mathcal{C}_{2*} L_{\mathbf{H}^{2n+1}})(\tilde{h}^{\frac{n}{2}} f).$$

Thus

$$(\mathcal{C}_{1*} L_{\mathbf{H}^{2n+1}}) f = n^{-(\frac{n}{2}+1)}(\mathcal{C}_{2*} L_{\mathbf{H}^{2n+1}})\left(n^{\frac{n}{2}} f\right),$$

where $n = \frac{\tilde{h}}{h}$. Note that $(\mathcal{C}_2)^* n = N^{-1}$, we have for any $F = \mathcal{C}_2^* f$,

$$(\mathcal{K}_* L_{\mathbf{H}^{2n+1}})F = N^{\frac{n}{2}+1} L_{\mathbf{H}^{2n+1}}(N^{-\frac{n}{2}} F).$$

□

Now we are ready to prove the relation between the inversion of Brownian motion on $\mathbf{H}^{2n+1}$ and the time changed Brownian bridge on $\mathbf{H}^{2n+1}$.

Proof of Theorem 1.2 Note that $N^{-\frac{n}{2}}$ is the Green function of the sub-Laplacian $L_{\mathbf{H}^{2n+1}}$ with pole $(0,0)$. We let

$$L^N F := L_{\mathbf{H}^{2n+1}} F + 2N^{\frac{n}{2}} \Gamma_{\mathbf{H}^{2n+1}}(N^{-\frac{n}{2}}, F), \tag{3.21}$$

where $\Gamma_{\mathbf{H}^{2n+1}}(F,G) = \frac{1}{2}(L_{\mathbf{H}^{2n+1}}(FG) - f L_{\mathbf{H}^{2n+1}} G - G L_{\mathbf{H}^{2n+1}} F)$ for any $F, G \in D_H$. From the previous theorem we have

$$\mathcal{K}_* L_{\mathbf{H}^{2n+1}} = N L^N.$$

Let X_t^N and X_t be Markov processes generated by $\frac{1}{2} L^N$ and $\frac{1}{2} L_{\mathbf{H}^{2n+1}}$. We first prove that X_t^N is X_t conditioned to be at the origin.

It suffices to prove that for any $F \in D_H$,

$$\mathbb{E}_x\left[F(X_t^N)\right] = \mathbb{E}_x\left[F(X_t)\mathbb{1}_{t<T} | X_\infty = (0,0)\right] \tag{3.22}$$

Let P_t^N and P_t be the heat semigroups generated by L^N and $L_{\mathbf{H}^{2n+1}}$ respectively, then by iterating (3.21) it is not hard to obtain

$$P_t^N(F(x)) = N(x)^{-\frac{n}{2}} P_t(N(x)^{-\frac{n}{2}} F(x)),$$

that is

$$\mathbb{E}_x\left[F(X_t^N)\right] = \frac{1}{N(x)^{-\frac{n}{2}}} \mathbb{E}_x\left[N(X_t)^{-\frac{n}{2}} F(X_t)\right] = \mathbb{E}_x\left[\frac{N(X_t)^{-\frac{n}{2}}}{N(x)^{-\frac{n}{2}}} F(X_t)\right].$$

From (3.22), we just need to show that

$$\mathbb{E}_x\left[F(X_t) | X_\infty = 0\right] = \mathbb{E}_x\left[\frac{N(X_t)^{-\frac{n}{2}}}{N^{-\frac{n}{2}}(x)} F(X_t)\right].$$

This is an easy consequence of $\mathbb{E}_x\left[X_\infty = 0\right] = N^{-\frac{n}{2}}(x)$ and

$$\mathbb{E}_x\left[F(X_t)\mathbb{1}_{X_\infty=0}\right] = \mathbb{E}_x\left[F(X_t)\mathbb{E}_x[\mathbb{1}_{X_\infty=0} | \mathcal{F}_t]\right] = \mathbb{E}_x\left[N(X_t)^{-\frac{n}{2}} F(X_t)\right].$$

Next we prove the time change. Consider the Markov process generated by $\frac{1}{2}\mathcal{K}_*(L_{\mathbf{H}^{2n+1}})$. It is the image of X_t under Kelvin transform, namely $\mathcal{K}(X_t)$. We claim

$$\mathcal{K}(X_t) = X_{\mathcal{A}_t}^N \tag{3.23}$$

where $\mathcal{A}_t = \int_0^t N(X_s)ds$ is the time-change of X^N. For any $F \in D_H$, we consider the associated martingale

$$M_t^F := F(X_t) - \frac{1}{2}\int_0^t L_{\mathbf{H}^{2n+1}} F(X_s)ds.$$

Denote $\tilde{F} = (\mathcal{K})^* F$. By plugging in (3.23), we obtain

$$M_t^F = \tilde{F}(X^N_{\mathcal{A}_t}) - \frac{1}{2}\int_0^t (L_{\mathbf{H}^{2n+1}} F) \circ \mathcal{K}^{-1}(X^h_{\mathcal{A}_s})ds = \tilde{F}(X^N_{\mathcal{A}_t}) - \frac{1}{2}\int_0^t N(X_s) L^N \tilde{F}(X^N_{\mathcal{A}_s})ds.$$

Let σ_t be the hitting time such that $\sigma_t = \inf\{u, \mathcal{A}_u > t\}$, then clearly $\mathcal{A}_{\sigma_t} = t = \sigma_{\mathcal{A}_t}$. By changing variable $s = \sigma_u$ we have

$$M_t^F = \tilde{F}(X^N_{\mathcal{A}_t}) - \frac{1}{2}\int_0^{\sigma_{\mathcal{A}_t}} N(X_s) L^N \tilde{F}(X^N_{\mathcal{A}_s})ds = \tilde{F}(X^N_t) - \frac{1}{2}\int_0^{\mathcal{A}_t} N(X_{\sigma_u}) L^N \tilde{F}(X^N_u)\sigma'_u du.$$

Note for any $u > 0$ we have $u = \mathcal{A}_{\sigma_u} = \int_0^{\sigma_u} N(X_s)ds$. By differentiating both sides with respect to u we obtain

$$1 = N(X_{\sigma_u})\sigma'_u.$$

Hence

$$M_t^F = \tilde{F}(X^N_t) - \frac{1}{2}\int_0^{\mathcal{A}_t} L^N \tilde{F}(X^N_u)du,$$

and we have the conclusion.

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