Math 461 Final, Fall 2005

Calculators, books, notes and extra papers are *not* allowed on this test!

Show all work to qualify for full credits

1. (9 points) (a) Suppose that A_1, A_2, A_3 and A_4 are independent events with $P(A_1) = 1/2$, $P(A_2) = 1/3$, $P(A_3) = 1/4$ and $P(A_4) = 1/5$. Find $P((A_1 \cup A_2) \cap (A_3 \cup A_4))$.

$$
\begin{aligned}
&P((A_1 \cup A_2) \cap (A_3 \cup A_4)) = P(A_1 \cup A_2)P(A_3 \cup A_4) \\
&= (P(A_1) + P(A_2) - P(A_1 \cap A_2))(P(A_3) + P(A_4) - P(A_3 \cap A_4)) \\
&= (\frac{1}{2} + \frac{1}{3} - \frac{1}{2} \cdot \frac{1}{3})(\frac{1}{4} + \frac{1}{5} - \frac{1}{4} \cdot \frac{1}{5}) \\
&= \frac{2}{3}\frac{2}{5} = \frac{4}{15}.
\end{aligned}
$$

(b) Suppose that X and Y are independent and identically distributed random variables with mean μ and variance σ^2, find $E[(X - 2Y)^2]$.

$$
\begin{aligned}
&E[(X - 2Y)^2] = E[X^2] - 4E[XY] + 4E[Y^2] = 5E[X^2] - 4[EX]^2 \\
&= 5(E[X^2] - [EX]^2) + [EX]^2 = 5\sigma^2 + \mu^2.
\end{aligned}
$$

(c) Suppose that X, Y and Z are independent random variables, X is Poisson with parameter $\lambda_1 = 1$, Y is geometric with parameter $p = 1/3$, and Z is exponential with parameter $\lambda_2 = 2$. Find $\mathrm{Cov}(X - 2Y + Z, 2X - 3Y - 2Z)$.

$$
\begin{aligned}
&\mathrm{Cov}(X - 2Y + Z, 2X - 3Y - 2Z) = \mathrm{Cov}(X, 2X) + \mathrm{Cov}(-2Y, -3Y) + \mathrm{Cov}(Z, -2Z) \\
&= 2\mathrm{Var}(X) + 6\mathrm{Var}(Y) - 2\mathrm{Var}(Z) \\
&= 2 \cdot 1 + 6 \cdot 6 - 2\frac{1}{4} = 37.5.
\end{aligned}
$$

2. (6 points) A closet has 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be exactly one complete pair?

$$
\frac{10 \cdot \binom{9}{6} \cdot 2^6}{\binom{20}{8}}.
$$

3. (7 points) If 5 married couples are lined up at random in a straight line, find the probability that no wife is next to her husband.

For $i = 1, \ldots, 5$, let E_i be the even that the i-th couple are together. Then the probability we are looking for is $1 - P(\cup_{i=1}^5 E_i)$. By the inclusion-exclusion formula we know that

$$\begin{aligned} P(\cup_{i=1}^5 E_i) &= \sum_{i=1}^5 P(E_i) - \sum_{i_1<i_2} P(E_{i_1}E_{i_2}) + \sum_{i_1<i_2<i_3} P(E_{i_1}E_{i_2}E_{i_3}) \\ &- \sum_{i_1<i_2<i_3<i_4} P(E_{i_1}E_{i_2}E_{i_3}E_{i_4}) + P(\cap_{i=1}^5 E_i) \\ &= 5 \cdot \frac{2 \cdot 9!}{(10)!} - \binom{5}{2} \cdot \frac{2^2 \cdot 8!}{(10)!} + \binom{5}{3} \cdot \frac{2^3 \cdot 7!}{(10)!} - \binom{5}{4} \cdot \frac{2^4 \cdot 6!}{(10)!} + \frac{2^5 \cdot 5!}{(10)!}. \end{aligned}$$

So the desired anwser is

$$1 - 5 \cdot \frac{2 \cdot 9!}{(10)!} + \binom{5}{2} \cdot \frac{2^2 \cdot 8!}{(10)!} - \binom{5}{3} \cdot \frac{2^3 \cdot 7!}{(10)!} + \binom{5}{4} \cdot \frac{2^4 \cdot 6!}{(10)!} - \frac{2^5 \cdot 5!}{(10)!}.$$

4. (7 points) A box initially contains 5 white and 7 black balls. Each time a ball is selected, its color is noted and it is replaced in the box along with 2 other balls of the same color. Find the probability that of the first three balls, exactly 2 are black.

With the obvious notation, the probability we are looking for is equal to $P(W_1B_2B_3) + P(B_1W_2B_3) + P(B_1B_2W_3)$.

$$\begin{aligned} P(W_1B_2B_3) &= \frac{5}{12}\frac{7}{14}\frac{9}{16} \\ P(B_1W_2B_3) &= \frac{7}{12}\frac{5}{14}\frac{9}{16} \\ P(B_1B_2W_3) &= \frac{7}{12}\frac{9}{14}\frac{5}{16} \end{aligned}$$

So the desired probability is

$$\frac{5 \cdot 7 \cdot 9}{4 \cdot 14 \cdot 16}.$$

5. (7 points) Independent trials, each results in a success with probability p and failure with probability $1 - p$, are performed. Let X be the total number of failures before the 5th success. Find the mass function of X.

X is a nonnegative integer-valued random variable. For any $x = 0, 1 \ldots$,

$$P(X = x) = \binom{4+x}{4} p^5 (1-p)^x.$$

6. (6 points) Suppose that X is uniformly distributed in $(-2,2)$. Find the density of the random variable $Y = X^2$.

Y takes values in $[0,4)$. For any $y \in (0,4)$,

$$P(Y \le y) = P(X^2 \le y) = P(-y^{1/2} \le X \le y^{1/2}) = \frac{y^{1/2}}{2}.$$

Thus the density of y is given by

$$g(y) = \begin{cases} \frac{1}{4}y^{-1/2}, & y \in (0,4) \\ 0, & \text{otherwise} \end{cases}$$

7. (8 points) Suppose that X_1 and X_2 are independent Poisson random variables with parameters $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Find the probability $P(X_1 = 40 | X_1 + X_2 = 100)$.

$X_1 + X_2$ is a Poisson random variable with parameter 3. So

$$\begin{aligned} P(X_1 = 40 | X_1 + X_2 = 100) &= \frac{P(X_1 = 40, X_1 + X_2 = 100)}{P(X_1 + X_2 = 100)} \\ &= \frac{P(X_1 = 40, X_2 = 60)}{P(X_1 + X_2 = 100)} \\ &= \frac{P(X_1 = 40)P(X_2 = 60)}{P(X_1 + X_2 = 100)} \\ &= \binom{100}{40} (\frac{1}{3})^{40} (\frac{2}{3})^{60}. \end{aligned}$$

8. (8 points) Suppose the joint density of X and Y is given by

$$f(x,y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 < x < 1, 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find $\text{Cov}(X,Y)$.

$$\begin{aligned} E[X] &= \int_0^1 \int_0^1 x\frac{3}{2}(x^2 + y^2)dxdy = \frac{5}{8} \\ E[Y] &= \int_0^1 \int_0^1 y\frac{3}{2}(x^2 + y^2)dxdy = \frac{5}{8} \\ E[XY] &= \int_0^1 \int_0^1 xy\frac{3}{2}(x^2 + y^2)dxdy = \frac{3}{8} \end{aligned}$$

Therefore

$$\text{Cov}(X,Y) = \frac{3}{8} - (\frac{5}{8})^2.$$

9. (8 points) Suppose that X and Y are independent random variables. If X is an exponential random variable with parameter $\lambda_1 = 1$ and Y is an exponential random variable with parameter $\lambda_2 = 2$, find the probability $P(X \geq Y)$.

$$\begin{aligned} P(X \geq Y) &= \int_0^\infty \left(\int_y^\infty e^{-x} 2e^{-2y} dx \right) dy \\ &= \int_0^\infty 2e^{-3y} dy = \frac{2}{3}. \end{aligned}$$

10. (8 points) Let X and Y be independent random variables. Suppose that X is a geometric random variable with parameter $p_1 = \frac{1}{2}$ and Y is a geometric random variable with parameter $p_2 = \frac{1}{3}$. Find the probability $P(X = Y)$.

$$\begin{aligned} P(X = Y) &= \sum_{i=1}^\infty P(X = i, X = Y) = \sum_{i=1}^\infty P(X = i, Y = i) \\ &= \sum_{i=1}^\infty P(X = i) P(Y = i) = \sum_{i=1}^\infty \frac{1}{2}\left(\frac{1}{2}\right)^{i-1} \frac{1}{3}\left(\frac{2}{3}\right)^{i-1} \\ &= \frac{1}{6} \sum_{j=0}^\infty \left(\frac{1}{3}\right)^j = \frac{1}{6}\frac{3}{2} = \frac{1}{4}. \end{aligned}$$

11. (8 points) Suppose that U and V are independent random variables and both are uniformly distributed in $(0,1)$. Define $X = \min(U, V)$ and $Y = \max(U, V)$. For $y \in (0,1)$, find $E(X|Y = y)$.

The joint density of X and Y is given by

$$f(x, y) = \begin{cases} 2, & 0 < x < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

So the marginal density of Y is

$$f_Y(y) = \begin{cases} 2y, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

and consequently for $y \in (0,1)$,

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y}, & 0 < x < y \\ 0, & \text{otherwise.} \end{cases}$$

Therefore $E(X|Y = y) = \frac{y}{2}$.

12. (8 points) Both player A and B try 100 free throws. It is known that, on average, A will make 80 percent of his free throw attempts and B will make 90 percent of his free throw attempts. Use the central limit theorem to find the probability at least 175 of these 200 free throw attempts will be successful.

By the central limit theorem, the number of free throws X made by A is approximately normal with mean 80 and variance 16, and the number of free throws Y made by A is approximately normal with mean 90 and variance 9. So the total number of free throws $X+Y$ is approximately normal with mean 170 and variance 25. Thus

$$P(X+Y \geq 170) = P(X+Y \geq 174.5) = P(\frac{X+Y-170}{5} \geq \frac{4.5}{5}) = 1 - \Phi(.9) = .1841.$$

13. (10 points) For a group of 100 people, find the expected number of days of the year that are birthdays of exactly 4 people.

For $= 1, \cdots, 365$, let $X_i = 1$ if day i is the birthdays of exactly 4 people and $X_i = 0$ otherwise. We are looking for $E(\sum_{i=1}^{365} X_i$ Since

$$P(X_i = 1) = \frac{\binom{100}{4}(364)^{96}}{(365)^{100}},$$

The anwser we are looking for is

$$365 \cdot \frac{\binom{100}{4}(364)^{96}}{(365)^{100}}.$$